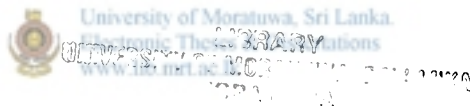


# CALCULATIONS ON FACE AND VERTEX REGULAR POLYHEDRA AND APPLICATION TO FINITE ELEMENT ANALYSIS

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This thesis was submitted to the Department of Mathematics of the University of Moratuwa in partial fulfillment of the requirements for the degree of M.Sc by research

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## DECLARATION

Work included in this thesis in part or whole, has not been submitted for any other academic qualification at any institution

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( Prof. G. T. F. de Silva, Supervisor )



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## LIST OF SYMBOLS

- $A_a$  -structure formed by bringing together  $a$  number of objects of type  $A$
- $A_a B_b \dots$  -structure formed by bringing together  $a$  number of objects of type  $A$ ,  
 $b$  number of objects of type  $B$ , .....
- $n_i$  -number of sides of the  $i$  th type polygon or the polygon of  $n_i$  number of sides
- $r_i$  -radius of the escribed circle radius of  $i$ th type polygon
- $M_i$  -number of  $i$  th type polygons meet at a vertex
- $R_i$  -radius of the escribed sphere radius of  $i$  th type polygon
- $R$  - radius of the escribed sphere radius of polyhedra
- $k$  -constant of the polyhedon
- $a$  -length of an edge
- $F$  -number of faces
- $E$  -number of edges
- $V$  -number of vertices
- 2D-two dimensional
- 3D- three dimensional
- $V(x, y)$  -two variable Lagrange polynomial
- $V(x, y, z)$  -three variable Lagrange polynomial
- ${}^n C_r$  -number of non repetitive combinations of  $n$  objects with  $r$  at a time
- ${}^n H_r$  -number of repetitive combinations of  $n$  objects with  $r$  at a time
- $B^{-1}$  -shape matrix
- $P$  -coordinate set of nodes with respect to  $X, Y$  coordinates
- $P'$  -coordinate set of nodes with respect to  $x, y$  coordinates
- $f(X, Y)$  -raw vector of terms of the piecewise polynomial of two variables
- $f(P)$  -matrix formed by substituting coordinate set of nodes to the terms of the  
piecewise polynomial
- $A$  -column vector of coefficients



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## ABSTRACT

Polyhedron is a solid figure bounded by plane faces. Face and vertex regular polyhedra are the polyhedra whose faces are regular polygons and the arrangement of polygons around each vertex is identical. Here general equations to calculate the properties of the face and vertex regular polyhedra are developed. This includes equations for radius of the escribed sphere and internal solid angle of a vertex. Using these equations the radius of the escribed sphere of face and vertex regular polyhedra are found including that of Snub Cube and Snub Dodecahedron. It is also shown that sphere is a limiting case of a polyhedron.

As application to finite element analysis, approximating the boundary by the sides of the finite elements is proposed. Also a method of defining the Lagrange interpolating polynomial is proposed. 2D tessellations are filling of infinite plane using polygons and 3D tessellations are filling of infinite space using polyhedra. With the piecewise polynomial selected in the above manner it is shown that the only possible regular tessellations that can be used in finite elements are Equilateral Triangle and Square in 2D and Triangular Regular Prism and Cube in 3D. It is shown in general that “any polygon having two axis of symmetry with nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains the complete polynomial of degree two” and “any polyhedron having a polygonal face with two axis of symmetry and having six or more number of vertices with the nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains a two variable complete polynomial of degree two”.