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CALCULATIONS ON FACE AND VERTEX REGULAR POLYHEDRA AND APPLICATION TO FINITE ELEMENT ANALYSIS

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This thesis was submitted to the Department of Mathematics of the University of Moratuwa in partial fulfillment of the requirements for the degree of M.Sc by research

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MORATUWA
SRI LANKA

APRIL 2006

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DECLARATION

Work included in this thesis in part or whole, has not been submitted for any other academic qualification at any institution

UOM Verified Signature

(U. C. Jayatilake)

Certified by

UOM Verified Signature

(Prof. G. T. F. de Silva, Supervisor)

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ACKNOLEDGEMENT

This is a result of a subject I developed for the last eight years. I introduced the word "Face and Vertex Regular Polyhedra" and I was able to publish a paper in Mathematical Gazette in March 2005, on the topic "Calculations on Face and Vertex Regular Polyhedra". Many people helped and encouraged me during the past years. Firstly I would like to thank my supervisors Prof.G.T.F. de Silva and Prof.M.Indralingam for their valuable advice. I would also like to thank all my friends and teachers who encouraged me during that time. My special thanks goes to Mr. Hema Nalin Karunarathne who gave me the chance to display my findings in 9.05 Rupavahini program in 1999. I would like to thank Prof. G.T.F. de Silva again for organizing a departmental seminar to present my findings to the staff of the Department of Mathematics, university of Moratuwa in 2001.

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TABLE OF CONTENTS

University of Moratuwa, Sri Lanka.
Electronic Theses & Dissertations
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University of Moratuwa, Sri Lanka.
Electronic Theses & Dissertations
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LIST OF SYMBOLS

A_a -structure formed by bringing together a number of objects of type A

$A_a B_b ....$ -structure formed by bringing together a number of objects of type A ,

b number of objects of type B , ...

n_i -number of sides of the i th type polygon or the polygon of n_i number of sides

r_i -radius of the escribed circle radius of ith type polygon

M_i -number of i th type polygons meet at a vertex

R_i -radius of the escribed sphere radius of i th type polygon

R - radius of the escribed sphere radius of polyhedra

k -constant of the polyhedon

a -length of an edge

F -number of faces

E -number of edges

V -number of vertices

2D-two dimensional

3D- three dimensional

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$V(x,y)$ -two variable Lagrange polynomial

$V(x,y,z)$ -three variable Lagrange polynomial

${}^nC_r$ -number of non repetitive combinations of n objects with r at a time

${}^nH_r$ -number of repetitive combinations of n objects with r at a time

B^{-1} -shape matrix

P -coordinate set of nodes with respect to X,Y coordinates

P' -coordinate set of nodes with respect to x,y coordinates

$f(X,Y)$ -raw vector of terms of the piecewise polynomial of two variables

$f(P)$ -matrix formed by substituting coordinate set of nodes to the terms of the piecewise polynomial

A -column vector of coefficients

LIST OF FIGURES

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LIST OF TABLES

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ABSTRACT

Polyhedron is a solid figure bounded by plane faces. Face and vertex regular polyhedra are the polyhedra whose faces are regular polygons and the arrangement of polygons around each vertex is identical. Here general equations to calculate the properties of the face and vertex regular polyhedra are developed. This includes equations for radius of the escribed sphere and internal solid angle of a vertex. Using these equations the radius of the escribed sphere of face and vertex regular polyhedrda are found including that of Snub Cube and Snub Dodecahedron. It is also shown that sphere is a limiting case of a polyhedron.

As application to finite element analysis, approximating the boundary by the sides of the finite elements is proposed. Also a method of defining the Lagrange interpolating polynomial is proposed. 2D tessellations are filling of infinite plane using polygons and 3D tessellations are filling of infinite space using polyhedra. With the piecewise polynomial selected in the above manner it is shown that the only possible regular tessellations that can be used in finite elements are Equilateral Triangle and Square in 2D and Triangular Regular Prism and Cube in 3D. It is shown in general that "any polygon having two axis of symmetry with nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains the complete polynomial of degree two" and "any polyhedron having a polygonal face with two axis of symmetry and having six or more number of vertices with the nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains a two variable complete polynomial of degree two".

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CHAPTER 1

TESSELLATIONS AND POLYHEDRA

INTRODUCTION

Polygon is a convex planner figure with straight edges. Regular polygon is a polygon with equal sides and equal internal angles. Regular polygon will be the theme thought this thesis.

2D tessellations are filling of infinite plane using polygons.

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Polyhedron is the 3 dimensional version of polygon. They are 3D convex objects bounded by plane faces. Face and vertex regular polyhedra are the polyhedra whose faces are regular polygons and the arrangement of the polygons around each vertex is identical.

3D tessellations are filling of space using polyhedra.

1.1 REGULAR POLYGONS

Polygon is a convex planner figure with straight edges. Regular polygon is a polygon with equal sides and equal internal angles.

Here only the regular polygons are considered for the constructions.

There are infinitely many types of regular polygons.

1.2 2D TESSELLATIONS [3]

2D tessellations are filling of infinite plane using polygons. A necessary requirement is that the sum of vertex angles of polygons $= 2\pi$. Here we use only the regular polygons for filling and we keep the arrangement of polygons around each vertex identical. They can be categorized as follows.

1. Regular 2D Tessellations:

   Only one type of polygon is used. 3 types exists.

2. Semi-Regular 2D Tessellations:

   Different types of polygons are used. 8 types exists.

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1.2.1 REGULAR 2D TESSELLATIONS

1. 3_6

Note: Here 3_6 means that 6 Triangles (3 sides) meet at a vertex.

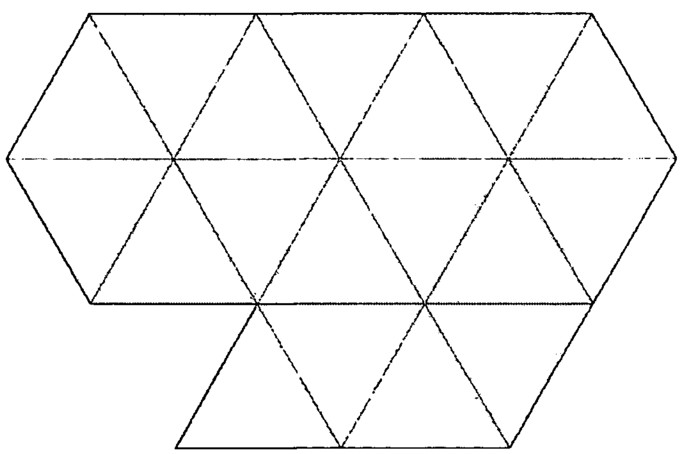

Figure 1.1

2. 4_4

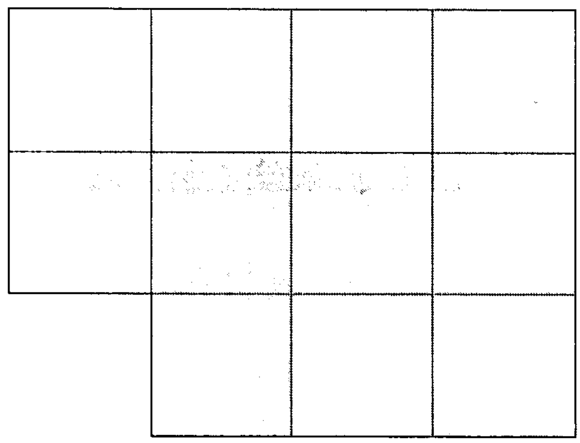

Figure 1.2

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3. 6_3

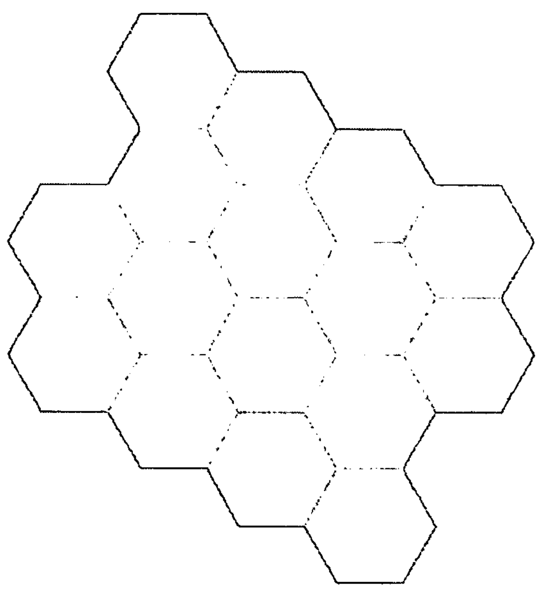

Figure 1.3

1.2.2 SEMI-REGULAR 2D TESSELLATIONS

1. $3_2 6_2$

Note: Here $3_2 6_2$ means that 2 Triangles (3 sides) and 2 Hexagons (6 sides) meet at a vertex.

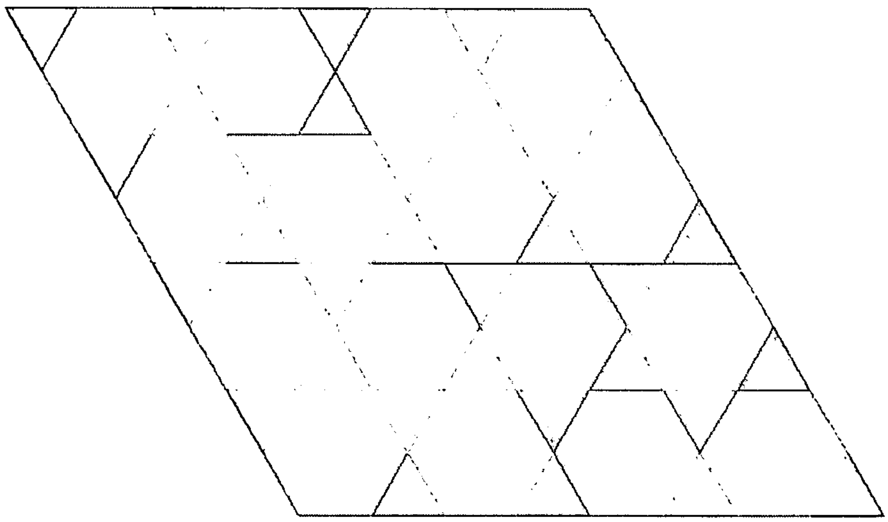

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Figure 1.4

2. $3_3 4_2$

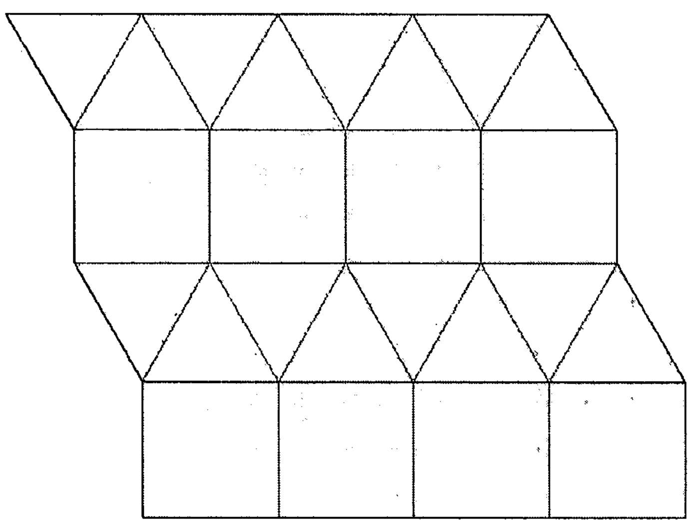

Figure 1.5

3. $3_2 4_1 3_1 4_1$ $\quad e.g: 3_2 4_1 3_1 4_1 (R)$

Note: There are Left hand(L) and Right hand(R) versions of this

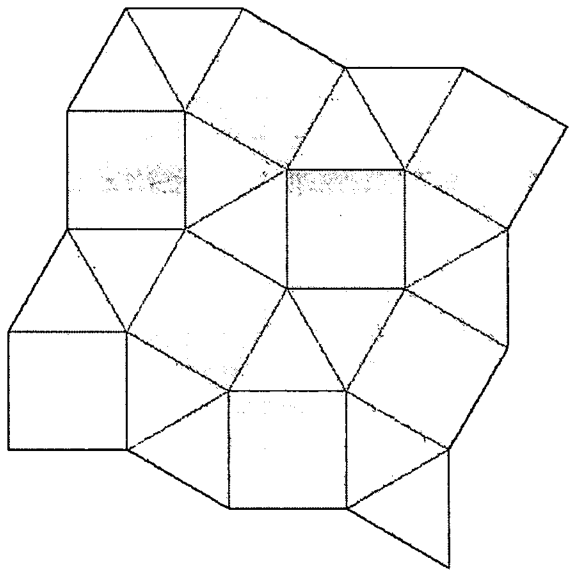

Figure 1.6

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4. $3_1 4_2 6_1$

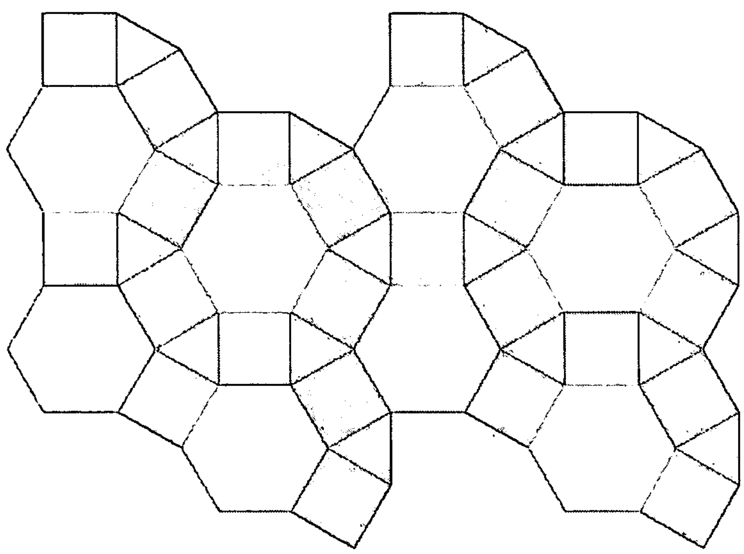

Figure 1.7

5. $3_1 12_2$

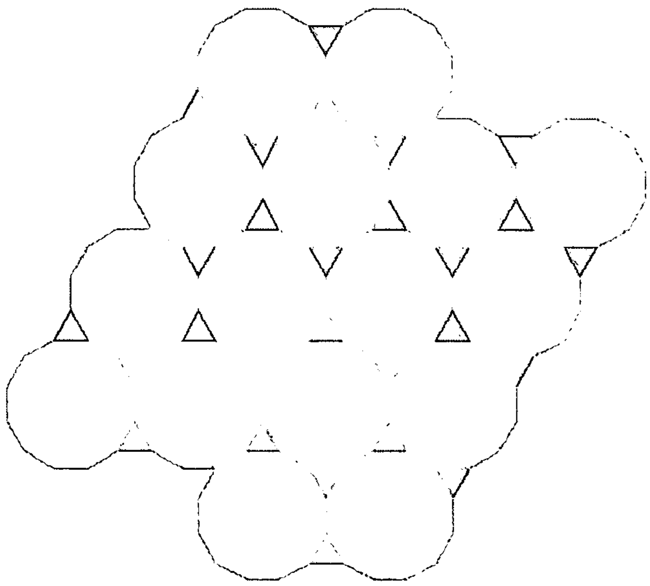

Figure 1.8

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6. $4_1 6_1 12_1$

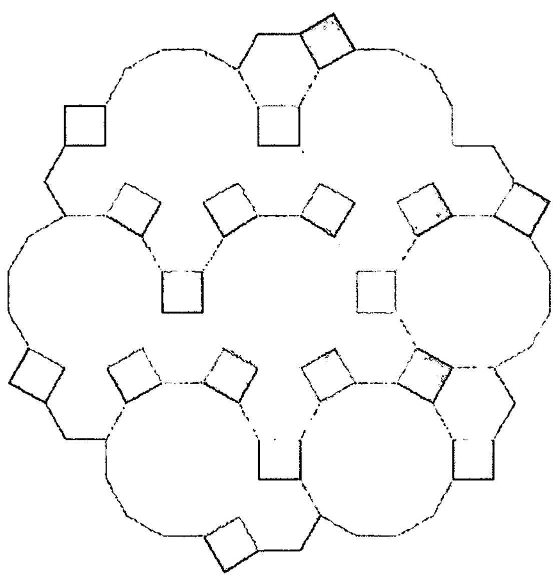

Figure 1.9

7. $4_1 8_2$

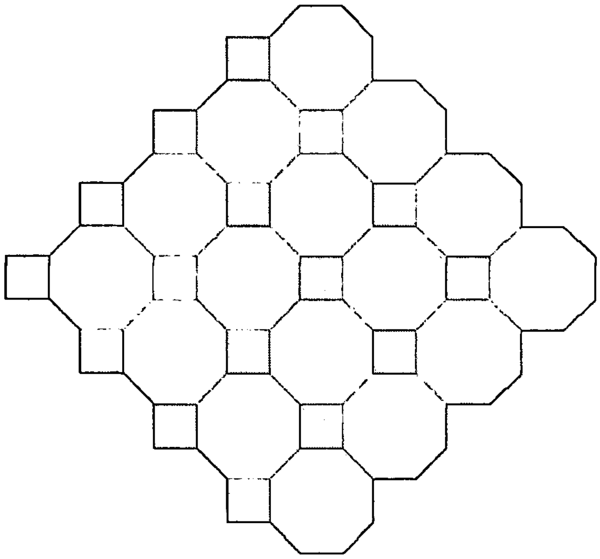

Figure 1.10

8. $3_4 6_1$ *e.g* : $3_4 6_1(R)$

Note: There are Left hand(L) and Right hand(R) versions of this

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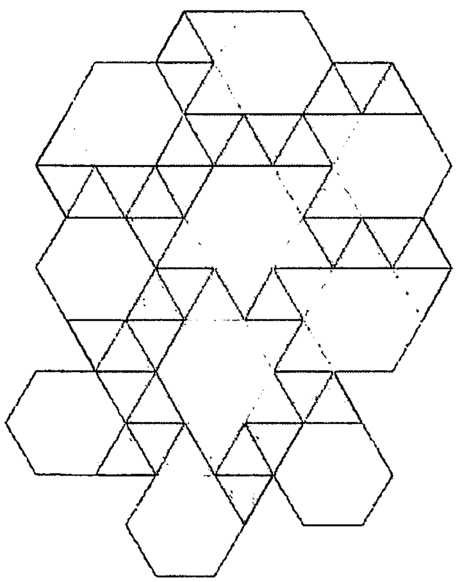

Figure 1.11

1.3 FACE AND VERTEX REGULAR POLYHEDRA

Polyhedron is the 3 dimensional version of polygon. They are 3D convex objects bounded by plane faces.

A necessary requirement is that the sum of vertex angles of polygons $< 2\pi$.

Face and vertex regular polyhedra are the polyhedra whose faces are regular polygons and the arrangement of the polygons around each vertex is identical. They can be categorized as

1. Regular Polyhedra (Platonic Solids):

    Only one type of polygon is used. 5 types exists.

2. Archimedean Polyhedra:

    Different types of polygons are used. 13 types exists.

3. Regular Prisms:

    Polygons are used for top and bottom with squares as sides. ∞ types exists.

4. Regular Anti-prisms:

    Polygons are used for top and bottom with triangles as sides.∞ types exists.

1.3.1 REGULAR POLYHEDRA (PLATONIC SOLIDS)

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1. 3_3 – *Tetrahedron*

Note: Here 3_3 means that 3 Triangles (3 sides) meet at a vertex.

This has 4 triangular faces, 4 vertices and 6 edges

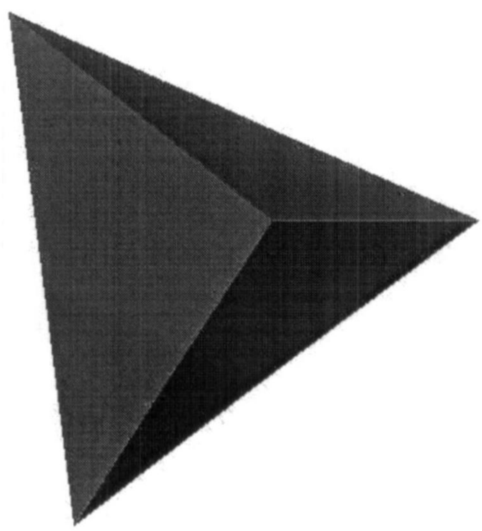

Figure 1.12

2. $4_3 - Hexahedron(Cube)$

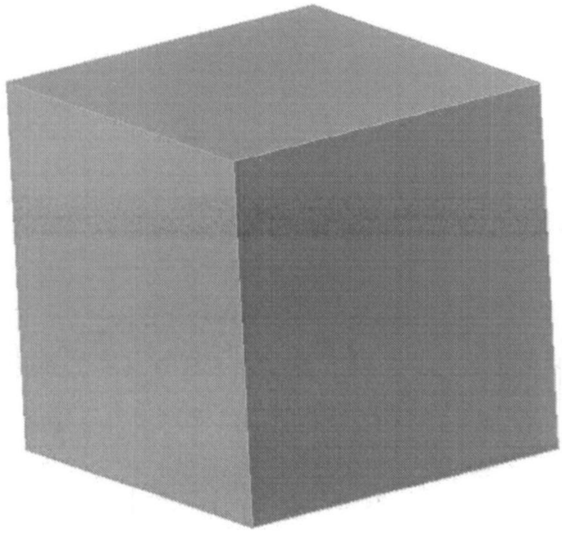

Figure 1.13

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3. $5_3 - Dodecahedron$

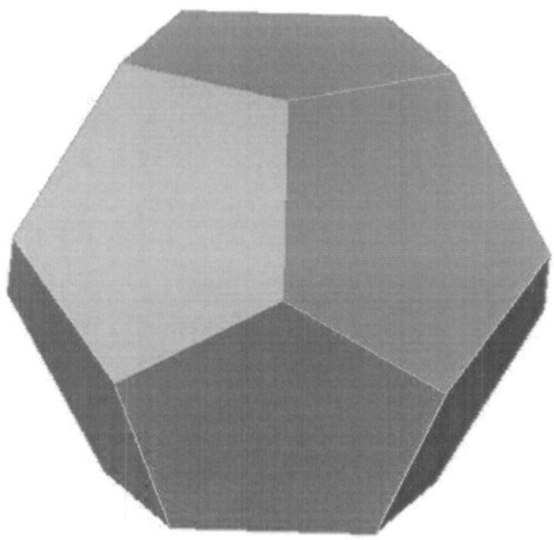

Figure 1.14

4. 3_4 – *Octahedron*

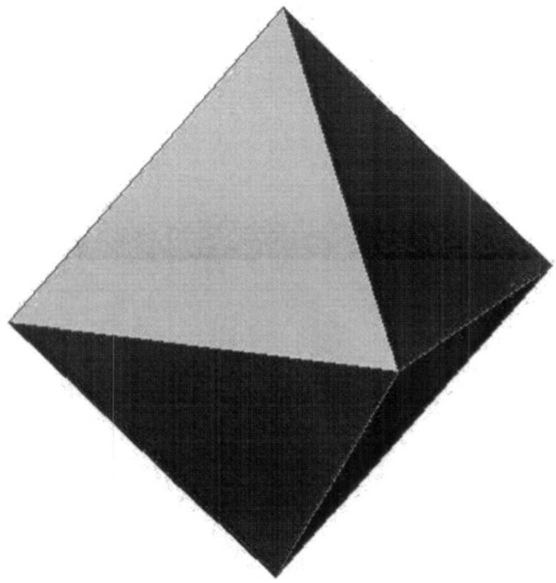

Figure 1.15

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5. 3_5 – *Icosahedron*

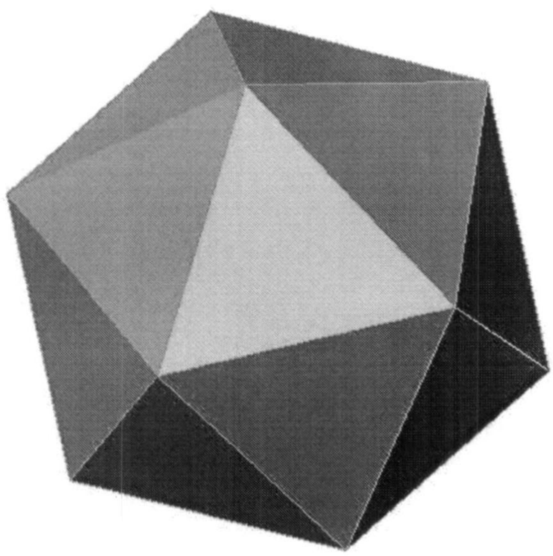

Figure 1.16

1.3.2 ARCHIMEDEAN POLYHEDRA

1. $3_1 6_2$ – *Truncated Tetrahedron*

Note: Here $3_1 6_2$ means that 1 Triangle (3 sides) and 2 Hexagons (6 Sides) meet at a vertex. This has 4 triangular and 4 hexagonal faces, 12 vertices and 18 edges

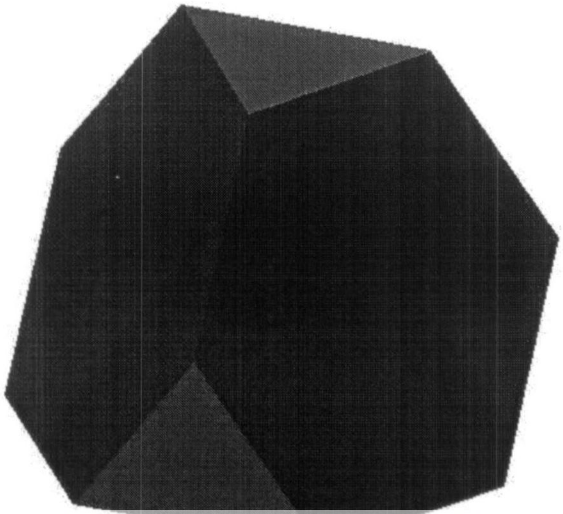

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Figure 1.17

2. $3_1 8_2$ – *Truncated Cube*

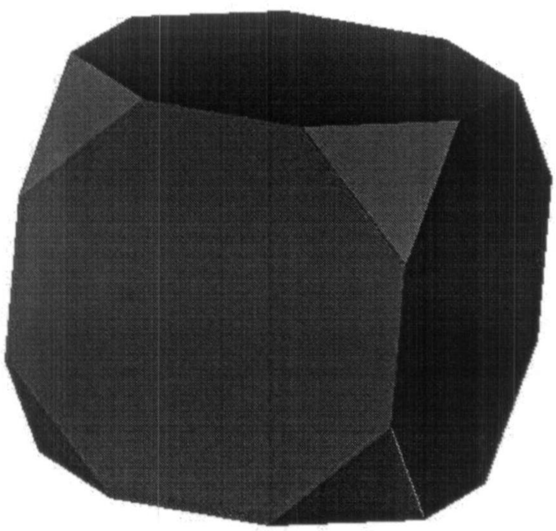

Figure 1.18

3. $3_1 10_2$ – *Truncated Dodecahedron*

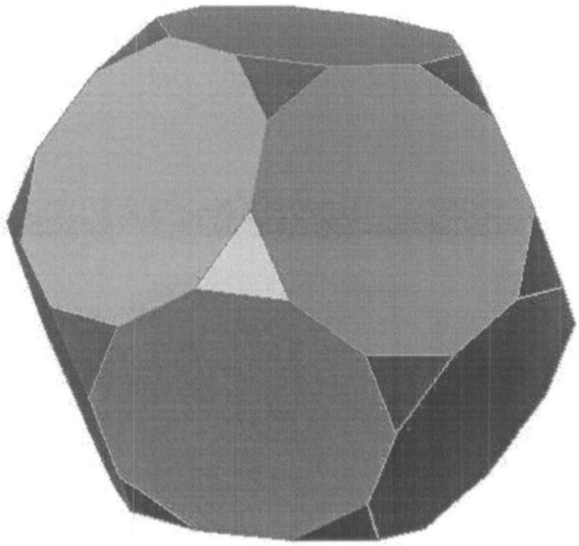

Figure 1.19

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4. $4_1 6_2$ – *Truncated Octahedron*

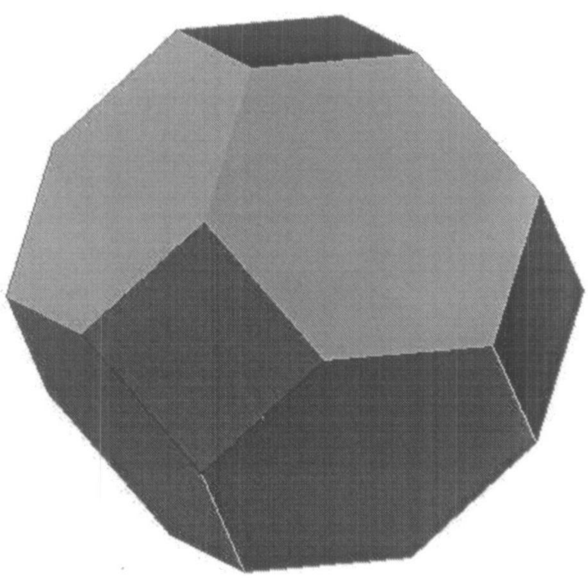

Figure 1.20

5. $5_1 6_2$ – *Truncated Icosahedron*

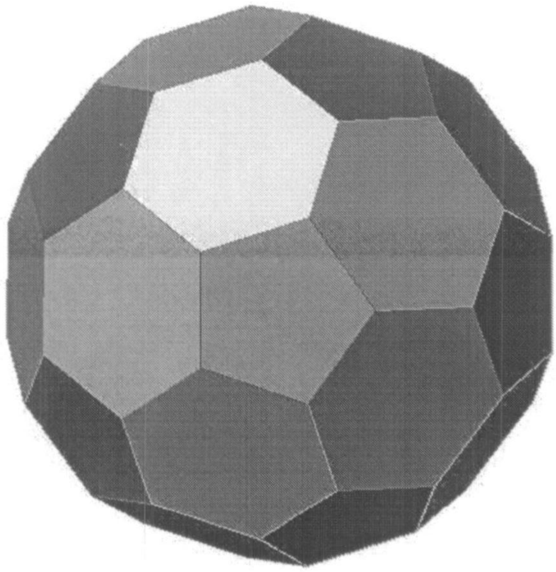

Figure 1.21

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6. $4_1 6_1 8_1$ – *Great Rhombicuboctahedron*

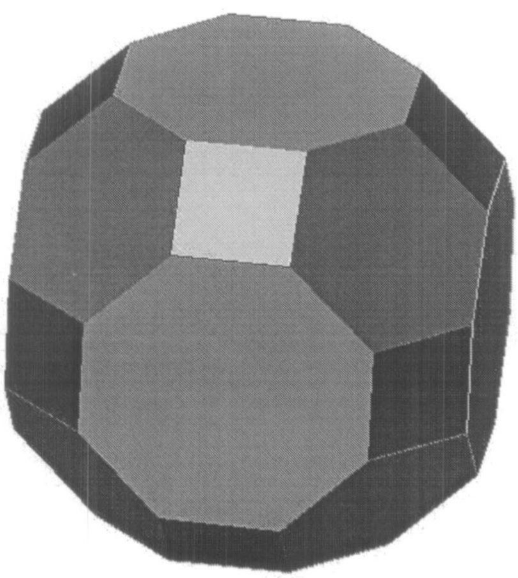

Figure 1.22

7. $4_1 6_1 10_1$ – *Great Rhombicosidodecahedron*

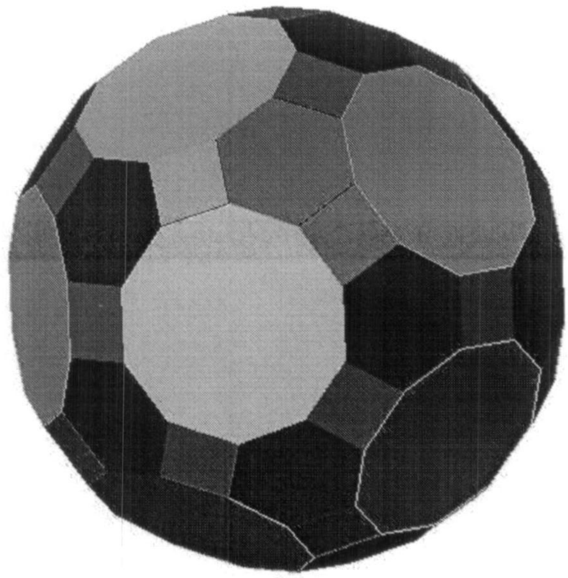

Figure 1.23

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8. $3_1 4_3$ – *Small Rhombicuboctahedron*

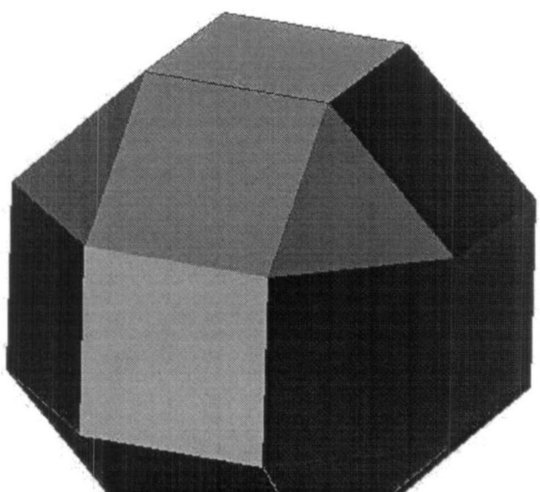

Figure 1.24

9. $3_2 4_2$ – *Cuboctahedron*

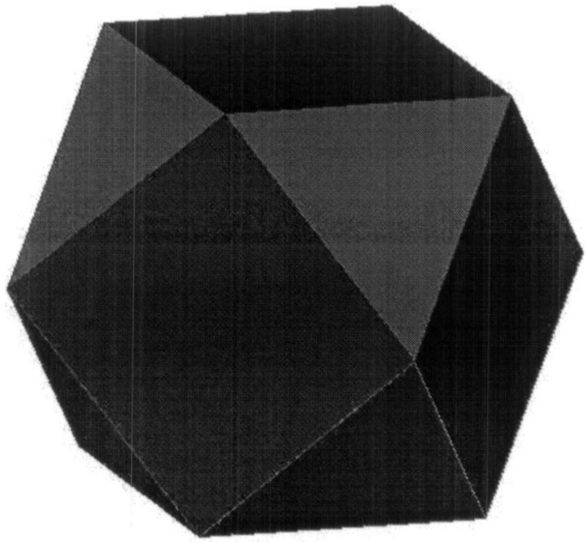

Figure 1.25

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10. $3_2 5_2$ – *Icosidodecahedron*

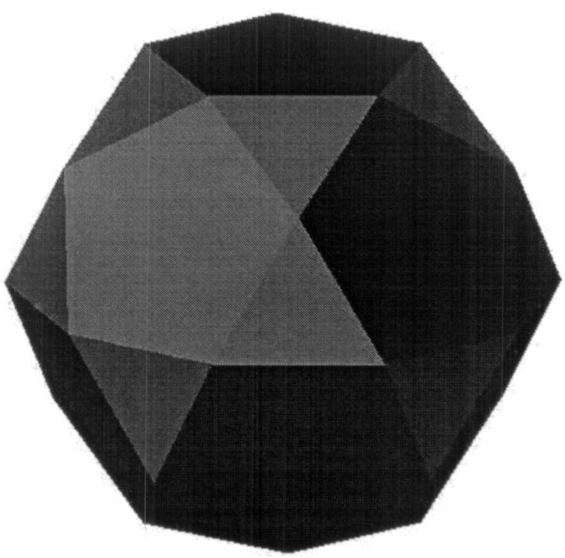

Figure 1.26

11. $3_1 4_2 5_1$ – *Small Rhombicosidodecahedron*

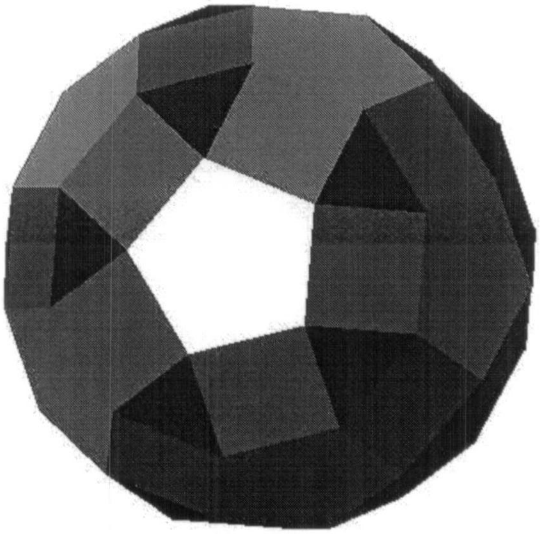

Figure 1.27

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12. $3_4 4_1$ – *Snub Cube* $e.g : 3_4 4_1 (L)$

Note: There are Left hand (L) and Right hand (R) versions of this

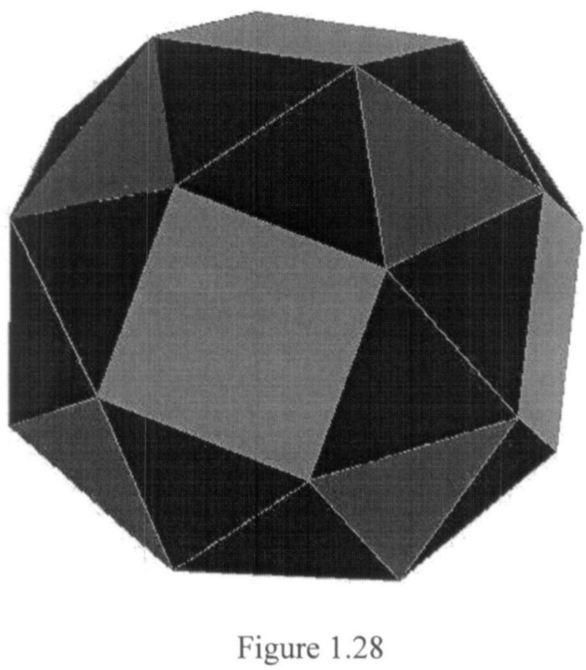

Figure 1.28

13. $3_4 5_1$ – *Snub Dodecahedron* $e.g: 3_4 5_1 (R)$

Note: There are Left hand (L) and Right hand (R) versions of this

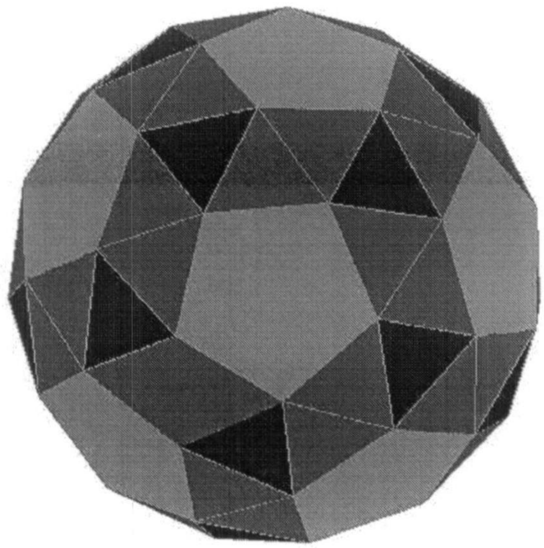

Figure 1.29

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1.3.3 REGULAR PRISMS

1. $n_1 4_2 ; n \neq 4$ $e.g: 4_2 6_1$ – *Hexagonal Regular Prism*

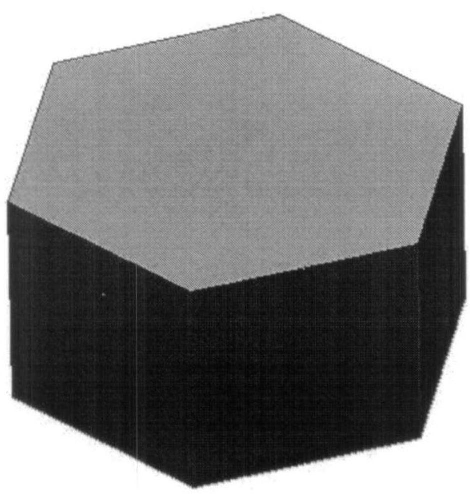

Figure 1.30

1.3.4 REGULAR ANTI-PRISMS

1. $n_1 3_3 \;; n \neq 3 \quad e.g: 3_3 6_1$ – *Hexagonal Regular AntiPrism*

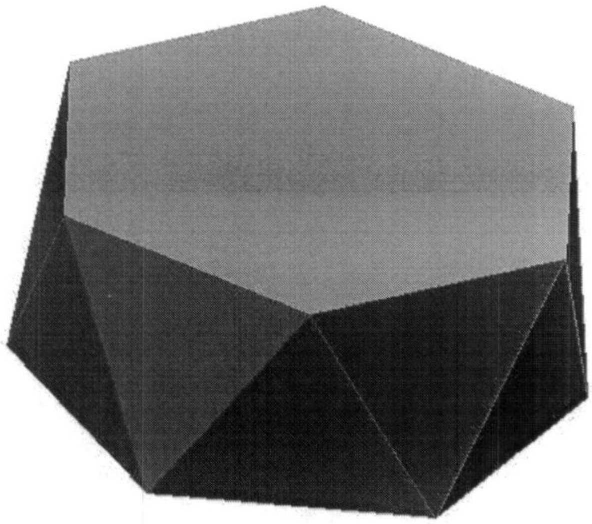

Figure 1.31

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1.4 3D TESSELLATIONS [4]

3D tessellations are filling of space using polyhedra.

A necessary requirement is that the sum of vertex solid angles of polyhedra $= 4\pi$.

Here we use only the face and vertex regular polyhedra for filling and we keep the arrangement of polyhedra around each vertex identical. They can be categorized as follows.

1. Regular 3D Tessellations:

    Only one type of Platonic and Archimedean Polyhedra are used. 2 types exists.

2. Regular Prism 3D Tessellations:

    Only one type of Regular Prism is used. 2 types exists.

3. Semi-Regular 3D Tessellations:

    Combinations of Polyhedra are used. 11 types exists.

4. Semi-Regular Prism 3D Tessellations:

    Different types of Regular Prisms are used. 8 types exists.

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1.4.1 REGULAR 3D TESSELLATIONS

1. $(4_3)_8$

Note: Here $(4_3)_8$ means that 8 cubes (4_3) meet at a vertex.

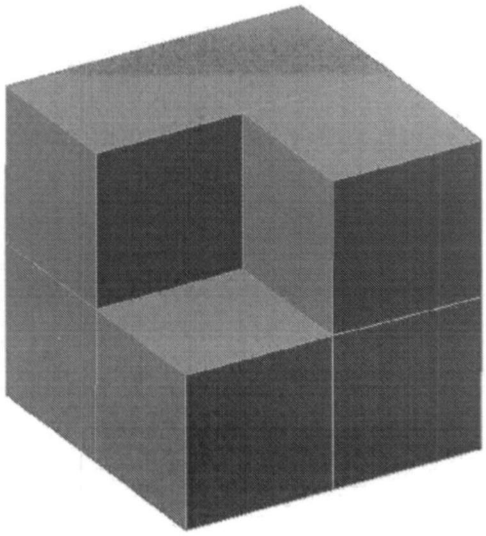

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Figure 1.32

2. $(4_1 6_2)_4$

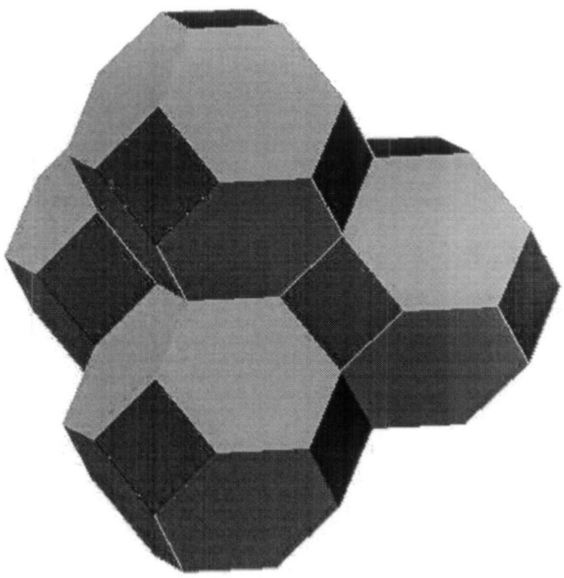

Figure 1.33

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1.4.2 REGULAR PRISM 3D TESSELLATIONS

1. $(3_1 4_2)_{12}$

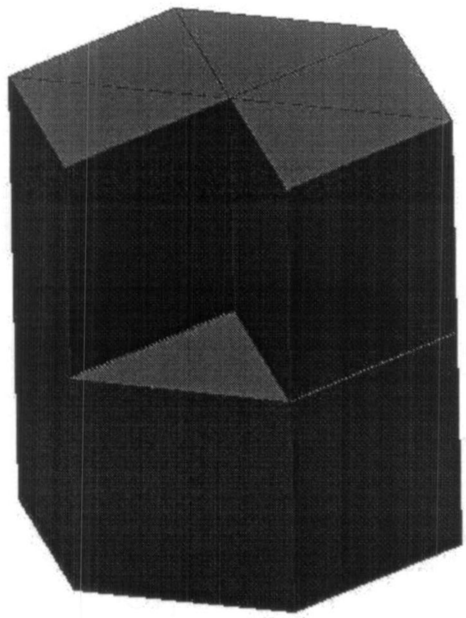

Figure 1.34

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2. $(6_1 4_2)_6$

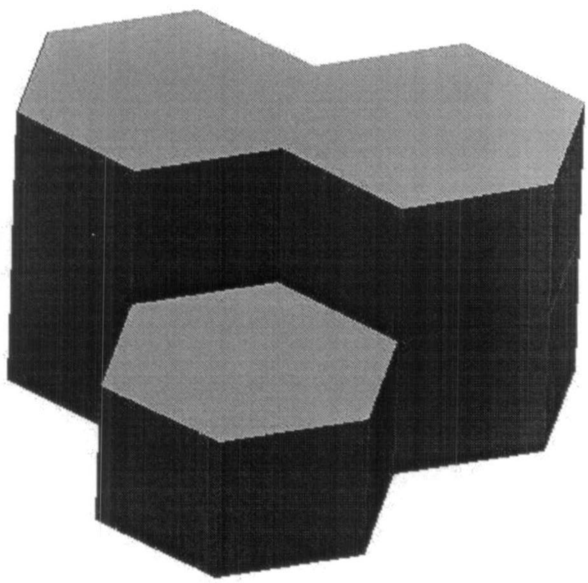

Figure 1.35

1.4.3 SEMI-REGULAR 3D TESSELLATIONS

1. $(3_3)_8(3_4)_6$

Note: Here $(3_3)_8$ means that 8 Tetradedra(3_3) and 6 Octahedra(3_4) meet at a vertex.

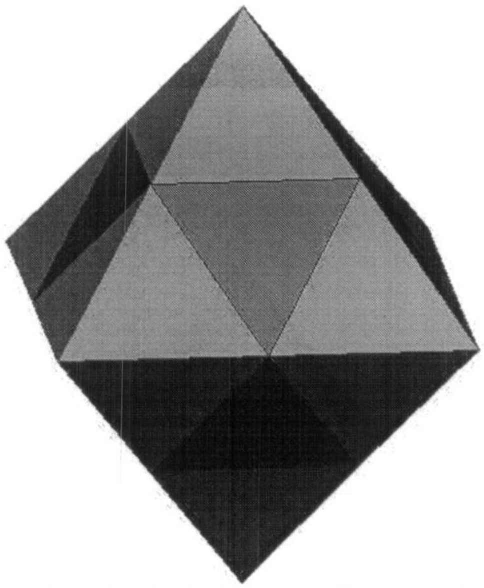

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Figure 1.36

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2. $(3_3)_2(3_1 6_2)_6$

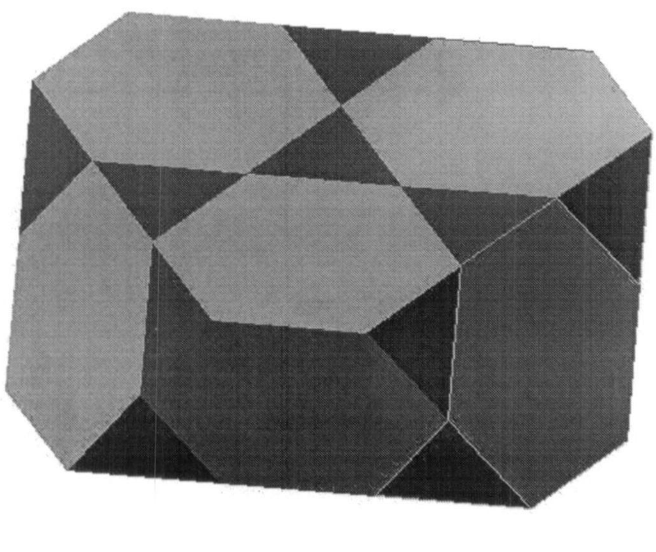

Figure 1.37

3. $(3_4)_1(3_1 8_2)_4$

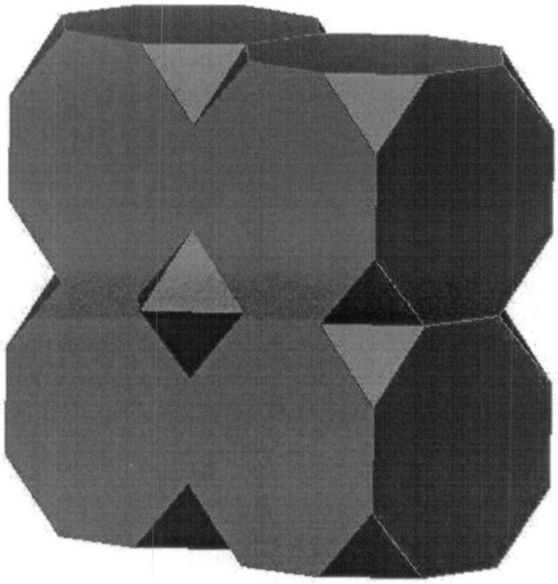

Figure 1.38

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4. $(3_4)_2(3_2 4_2)_4$

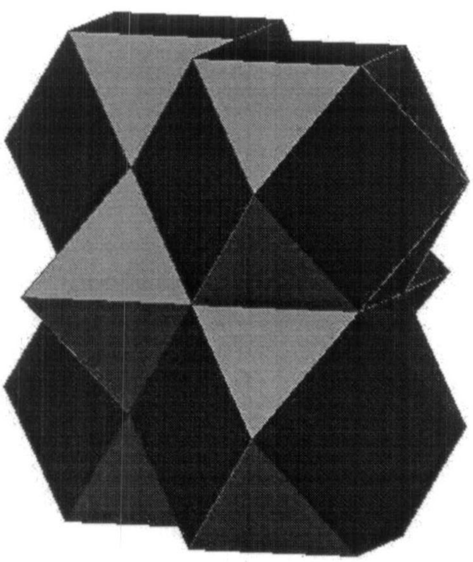

Figure 1.39

5. $(4_3)_1(4_16_2)_1(4_16_18_1)_2$

Figure 1.40

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6. $(3_16_2)_1(3_18_2)_1(4_16_18_1)_2$

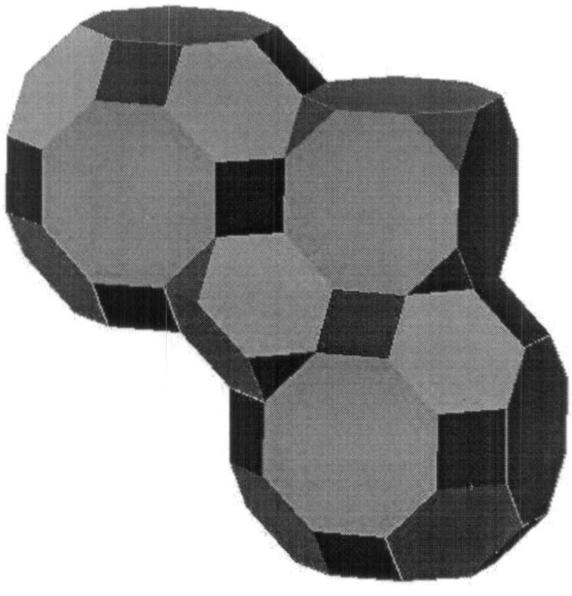

Figure 1.41

7. $(4_3)_2(3_24_2)_1(3_14_3)_2$

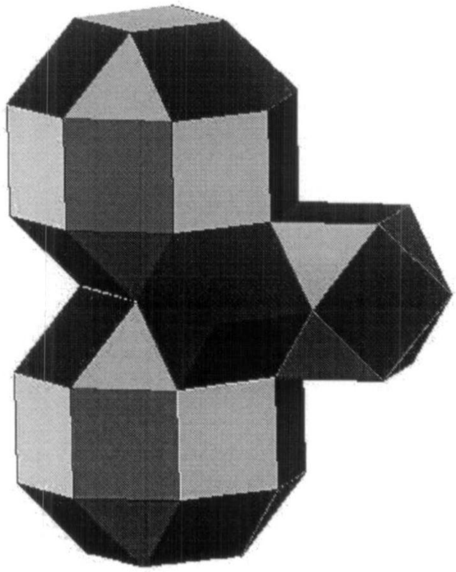

Figure 1.42

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8. $(3_3)_1(4_3)_1(3_14_3)_3$

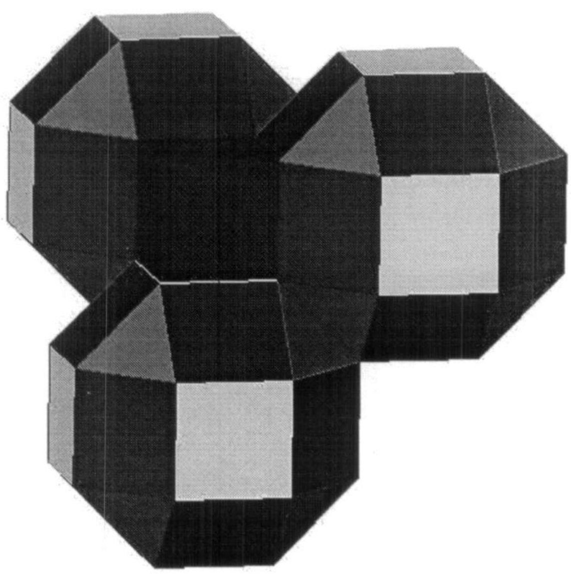

Figure 1.43

9. $(3_1 6_2)_2 (3_2 4_2)_1 (4_1 6_2)_2$

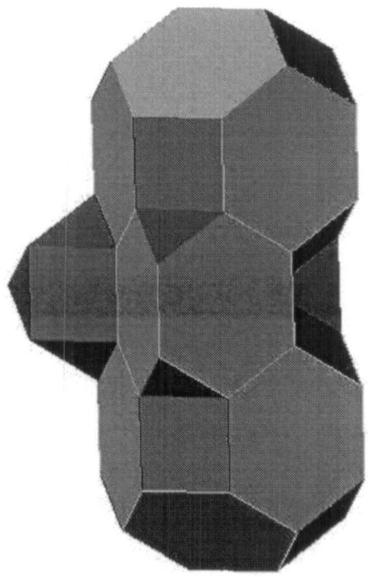

Figure 1.44

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10. $(4_2 8_1)_2 (4_1 6_1 8_1)_2$

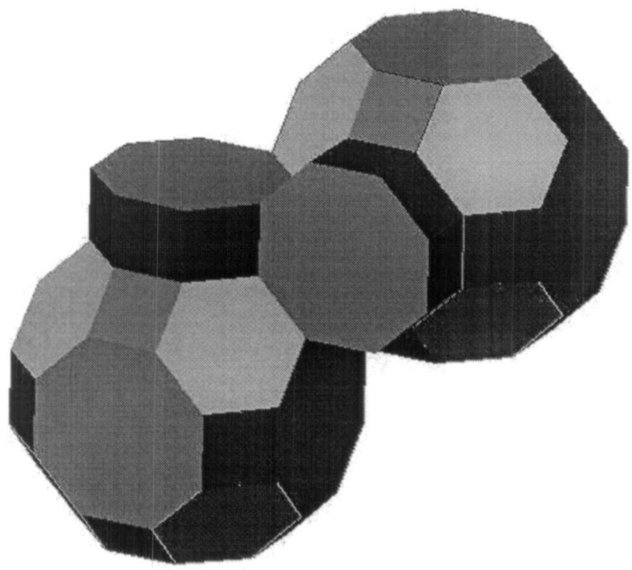

Figure 1.45

11. $(4_3)_1(4_28_1)_2(3_18_2)_1(3_14_3)_1$

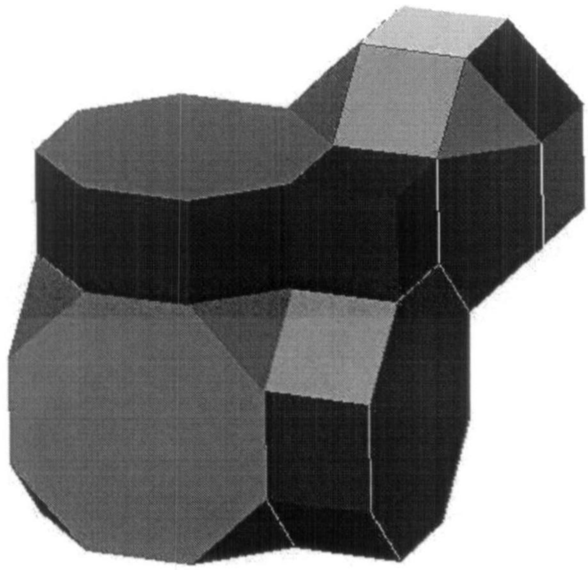

Figure 1.46

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1.4.4 SEMI-REGULAR PRISM 3D TESSELLATIONS

1. $(3_14_2)_4(6_14_2)_4$
2. $(3_14_2)_6(4_3)_4$
3. $(3_14_2)_4(4_3)_2(3_14_2)_2(4_3)_2$
4. $(3_14_2)_2(4_3)_4(6_14_2)_2$
5. $(3_14_2)_2(12_14_2)_4$
6. $(4_3)_2(6_14_2)_2(12_14_2)_2$
7. $(4_3)_2(8_14_2)_4$
8. $(3_14_2)_8(6_14_2)_2$

CHAPTER 2

CALCULATIONS ON FACE AND VERTEX REGULAR POLYHEDRA

INTRODUCTION

Due to the similarity of their vertices Face and Vertex Regular Polyhedra have a unique escribed sphere. Any geometrical property of the above Polyhedra can be found if this is known. Here equations will be developed to find the exact escribed radii values for Face and Vertex Regular Polyhedra.

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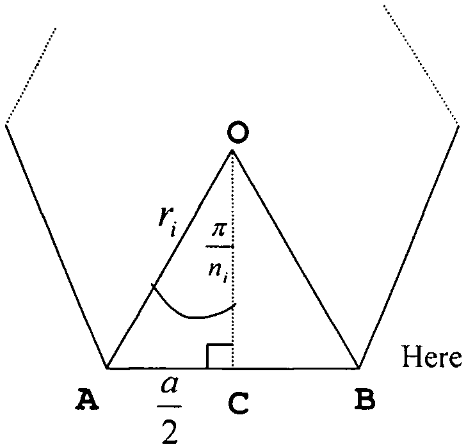
2.1 ESCRIBED RADIUS OF A FACE AND VERTEX REGULAR POLYHEDRON [1]

When the vertices of a regular polygon with n_i number of sides are joined to its center **O**, n_i number of equilateral triangle are formed as shown in figure 2.1.



$$A\hat{O}C = \frac{A\hat{O}B}{2} = \frac{\pi}{n_i}$$

$$\Rightarrow r_i = \frac{a}{2\sin\frac{\pi}{n_i}} \text{------} (1)$$

Here a =length of a side of the polygon

Figure 2.1

Suppose that this polygon is placed inside a sphere of radius R_i and center **G**. Then all the vertices will touch the surface of the sphere and the triangle **ABO** is seen as in the following figures.

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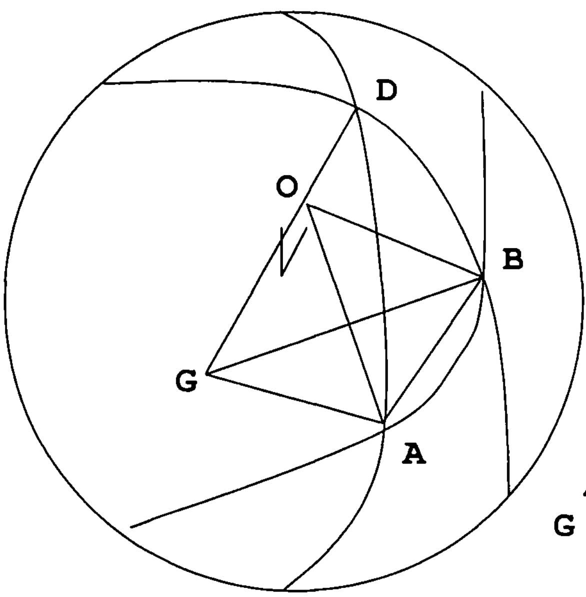

Figure 2.2

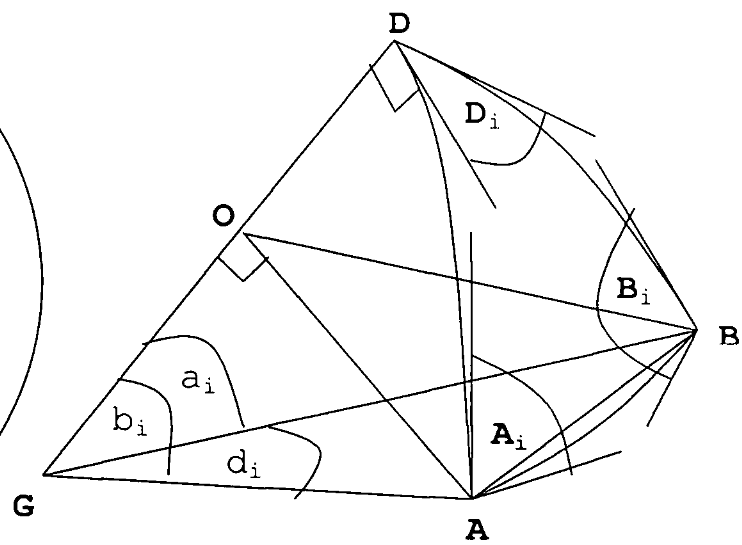

Figure 2.3

A spherical triangle **ABD** is formed when the triangle **ABO** is projected on to the surface of the sphere. Let the angles of the spherical triangle **ABD** be A_i, B_i and D_i. Also let the corresponding angles between lines joining to the center **G** be a_i, b_i and d_i respectively. All the angles around the point **D** will form a plane perpendicular to **GD** at the point **D**. Hence

$$n_i D_i = 2\pi \Rightarrow D_i = \frac{2\pi}{n_i} \text{------}(2)$$

OA=OB=a gives

$$A_i = B_i \text{------}(3)$$

$$a_i = b_i \text{------}(4)$$

The triangles **ABG** and **BGO** can be separated as follows

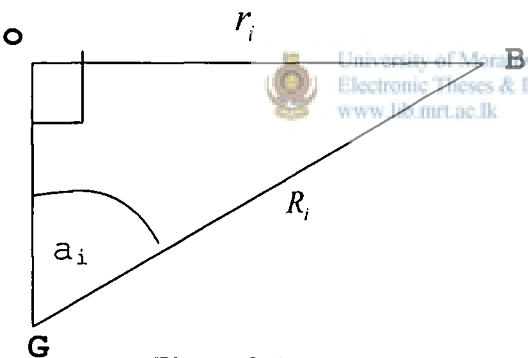

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Figure 2.4

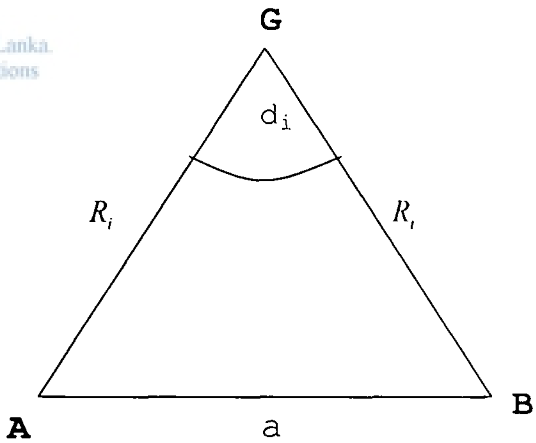

Figure 2.5

From figure 2.5

$$a^2 = R_i^2 + R_i^2 - 2R_i R_i \cos d_i$$

$$\Rightarrow \cos d_i = \frac{2R_i^2 - a^2}{2R_i^2}$$

$$\Rightarrow \sin d_i = \frac{a\sqrt{4R_i^2 - a^2}}{2R_i^2} \text{------}(5)$$

By figure 2.4 and equation(1)

$$\sin a_i = \frac{r_i}{R_i} = \frac{a}{2R_i \sin\frac{\pi}{n_i}} \quad -----(6)$$

But by a theorem in spherical trigonometry

$$\frac{\sin a_i}{\sin A_i} = \frac{\sin b_i}{\sin B_i} = \frac{\sin d_i}{\sin D_i} \qquad \text{[APPENDIX A]}$$

$(2),(5),(6) \Rightarrow$

$$\frac{\dfrac{a}{2R_i \sin\frac{\pi}{n_i}}}{\sin A_i} = \frac{\dfrac{a\sqrt{4R_i^{\,2} - a^2}}{2R_i^{\,2}}}{\sin\frac{2\pi}{n_i}}$$

$$\Rightarrow R_i = \frac{a}{2}\frac{1}{\sqrt{1 - \left(\dfrac{\cos\frac{\pi}{n_i}}{\sin A_i}\right)^2}}$$

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This is the radius of the escribed sphere.

Now suppose that different types of polygons are placed inside the sphere and the radius is adjusted in such a way that a 3D vertex(**A**) is formed with the adjacent sides of polygons are touching each other. At this position radii values calculated for different types of polygons are equal.i.e.

$$R_i = constant(R, say)$$

$$\Rightarrow \frac{\cos\frac{\pi}{n_i}}{\sin A_i} = constant(k, say) \quad -----(8)$$

So the escribed sphere radius is $R = \frac{a}{2}\frac{1}{\sqrt{1-k^2}}$ --------(7)

When 3D vertex is formed at **A**, sum of angles A_i will add up to 2π creating a plane perpendicular to **GA** at **A**.

If M_i number of polygons with n_i number of sides meet at the vertex **A**, and because each polygon provides two angles this result can be written as

$$\sum_i 2M_i A_i = 2\pi$$

$$\Rightarrow \sum_i M_i A_i = \pi \text{-----}(9)$$

To find the radius of the escribed sphere radius, R by (7) the value of the constant k must be found. To find k, equations (8) and (9) must be solved to eliminate A_i.

The equations (8) and (9) cannot be solved in closed form. But (7),(8) and (9) can be combined to give the following identity.

$$\sum_i M_i \sin^{-1}\left(\frac{\cos\frac{\pi}{n_i}}{\sqrt{1-\left(\frac{a}{2R}\right)^2}}\right) = \pi$$

Here a = length of an edge which is constant for the polyhedron.

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2.2 COMPUTATION OF ESCRIBED RADIUS

Due to the similarity of its vertices, face and vertex regular polyhedra have a unique escribed sphere.

Escribed radius can be computed by solving

$$\frac{\cos\frac{\pi}{n_i}}{\sin A_i} = constant = k \text{-----}(8)$$

$$\sum_i M_i A_i = \pi \text{-----}(9)$$

or

$$\frac{\cos\frac{\pi}{n_1}}{\sin A_1} = \frac{\cos\frac{\pi}{n_2}}{\sin A_2} = \frac{\cos\frac{\pi}{n_3}}{\sin A_3} = \ldots\ldots\ldots\ldots\ldots \text{-----}(8)$$

$$M_1A_1 + M_2A_2 + M_3A_3 + \ldots\ldots\ldots\ldots\ldots\ldots = \pi \text{-----}(9)$$

As stated early this system cannot be solved in closed form.

However for a given 3D vertex, (8) and (9) can be solved to find A_i and then k. By substituting it in (7) the radius of the escribed sphere can be found. Following illustrates how this can be done for the face and vertex regular polyhedra.

(1) For $3_3, 3_4, 3_5, 4_3, 5_3$ (Regular Polyhedra)

The vertex is of the form n_M. Then

$$MA = \pi \text{ ----------- } (8)$$

$$k = \frac{\cos\frac{\pi}{n}}{\sin A} \text{ ---------- } (9)$$

$$(8) \Rightarrow A = \frac{\pi}{M}$$

$$(9) \Rightarrow k = \frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{M}}$$

So the radius is $R = \dfrac{a}{2}\dfrac{1}{\sqrt{1-k^2}} = \dfrac{a}{2}\dfrac{1}{\sqrt{1-\left(\dfrac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{M}}\right)^2}}$

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The calculated exact escribed radius values are

$$R_{3_3} = \frac{\sqrt{6}}{4}a$$

$$R_{4_3} = \frac{\sqrt{3}}{2}a$$

$$R_{3_4} = \frac{\sqrt{2}}{2}a$$

$$R_{5_3} = \frac{\sqrt{6\left(3+\sqrt{5}\right)}}{4}a$$

$$R_{3_5} = \frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4}a$$

(2) For $3_1 6_2, 3_1 8_2, 3_1 10_2, 4_1 6_2, 5_1 6_2$

The vertex is of the form $n1_1 n2_2$. Then

$$A_1 + 2A_2 = \pi \text{ - - - - - - - - - - } (8)$$

$$\frac{\cos\frac{\pi}{n_1}}{\sin A_1} = \frac{\cos\frac{\pi}{n_2}}{\sin A_2} = k \text{ - - - - - - - - - } (9)$$

$$(8) \Rightarrow \sin A_1 = \sin(\pi - 2A_2) = \sin 2A_2 = 2\sin A_2 \cos A_2$$

$$(9) \Rightarrow \frac{\cos\frac{\pi}{n_1}}{\cos\frac{\pi}{n_2}} \sin A_2 = 2\sin A_2 \cos A_2$$

$$\Rightarrow \cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}}$$

The calculated exact escribed radius values are

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$$R_{3_1 6_2} = \frac{\sqrt{22}}{4} a$$

$$R_{3_1 8_2} = \frac{\sqrt{7 + 4\sqrt{2}}}{2} a$$

$$R_{3_1 10_2} = \frac{\sqrt{2(37 + 15\sqrt{5})}}{4} a$$

$$R_{4_1 6_2} = \frac{\sqrt{10}}{2} a$$

$$R_{5_1 6_2} = \frac{\sqrt{2(29 + 9\sqrt{5})}}{4} a$$

(3) For $4_16_18_1, 4_16_110_1$

The vertex is of the form $n1_1n2_1n3_1$. Then

$$A_1 + A_2 + A_3 = \pi ---------- (8)$$

$$\frac{\cos\frac{\pi}{n_1}}{\sin A_1} = \frac{\cos\frac{\pi}{n_2}}{\sin A_2} = \frac{\cos\frac{\pi}{n_3}}{\sin A_3} = k -------- (9)$$

$$(8) \Rightarrow \sin(A_1 + A_2) = \sin(\pi - A_3) = \sin A_3$$
$$\Rightarrow \sin A_1 \cos A_2 + \cos A_1 \sin A_2 = \sin A_3$$
$$(9) \Rightarrow \frac{1}{k}\cos\frac{\pi}{n_1}\cos A_2 + \cos A_1 \frac{1}{k}\cos\frac{\pi}{n_2} = \frac{1}{k}\cos\frac{\pi}{n_3}$$
$$\Rightarrow \left(\cos\frac{\pi}{n_3} - \cos A_1 \cos\frac{\pi}{n_2}\right)^2 = \left(\cos\frac{\pi}{n_1}\cos A_2\right)^2$$
$$\Rightarrow \cos^2\frac{\pi}{n_3} + \cos^2 A_1 \cos^2\frac{\pi}{n_2} - 2\cos A_1 \cos\frac{\pi}{n_2}\cos\frac{\pi}{n_3}$$
$$= \cos^2\frac{\pi}{n_1}\left(1 - \sin^2 A_2\right)$$
$$= \cos^2\frac{\pi}{n_1}\left(1 - \sin^2 A_1 \frac{\cos^2\frac{\pi}{n_2}}{\cos^2\frac{\pi}{n_1}}\right)$$
$$\Rightarrow \cos^2\frac{\pi}{n_3} + \cos^2\frac{\pi}{n_2} - \cos^2\frac{\pi}{n_1} = 2\cos A_1 \cos\frac{\pi}{n_2}\cos\frac{\pi}{n_3}$$
$$\Rightarrow \cos A_1 = \frac{\cos^2\frac{\pi}{n_3} + \cos^2\frac{\pi}{n_2} - \cos^2\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}\cos\frac{\pi}{n_3}}$$

The calculated exact escribed radius values are

$$R_{4_16_18_1} = \frac{\sqrt{13 + 6\sqrt{2}}}{2}a$$
$$R_{4_16_110_1} = \frac{\sqrt{31 + 12\sqrt{5}}}{2}a$$

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(4) For $3_1 4_3$

The vertex is of the form $n1_1 n2_3$. Then

$$A_1 + 3A_2 = \pi \text{ ----------(8)}$$

$$\frac{\cos\dfrac{\pi}{n_1}}{\sin A_1} = \frac{\cos\dfrac{\pi}{n_2}}{\sin A_2} = k \text{ ---------(9)}$$

$$(8) \Rightarrow \sin A_1 = \sin(\pi - 3A_2) = \sin 3A_2 = 3\sin A_2 - 4\sin^3 A_2$$

$$\Rightarrow \sin A_1 = \sin(\pi - 3A_2) = \sin 3A_2 = 3\sin A_2 - 4\sin^3 A_2$$

$$(9) \Rightarrow \frac{1}{k}\cos\frac{\pi}{n_1} = \frac{1}{k}\cos\frac{\pi}{n_2}\left(3 - 4\sin^2 A_2\right)$$

$$\Rightarrow 4\sin^2 A_2 = 3 - \frac{\cos\dfrac{\pi}{n_1}}{\cos\dfrac{\pi}{n_2}}$$

$$\Rightarrow \sin A_2 = \frac{1}{2}\sqrt{3 - \frac{\cos\dfrac{\pi}{n_1}}{\cos\dfrac{\pi}{n_2}}}$$

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The calculated exact escribed radius values are

$$R_{3_1 4_3} = \frac{\sqrt{5 + 2\sqrt{2}}}{2} a$$

(5) For $3_2 4_2, 3_2 5_2$

The vertex is of the form $n1_2 n2_2$. Then

$$2A_1 + 2A_2 = \pi \text{ ---------- } (8)$$

$$\frac{\cos\frac{\pi}{n_1}}{\sin A_1} = \frac{\cos\frac{\pi}{n_2}}{\sin A_2} = k \text{ -------- } (9)$$

$$(8) \Rightarrow \sin A_1 = \sin\left(\frac{\pi}{2} - A_2\right) = \cos A_2$$

$$(9) \Rightarrow \sin A_2 \frac{\cos\frac{\pi}{n_1}}{\cos\frac{\pi}{n_2}} = \cos A_2$$

$$\Rightarrow \tan A_2 = \frac{\cos\frac{\pi}{n_2}}{\cos\frac{\pi}{n_1}}$$

The calculated exact escribed radius values are

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$$R_{3_2 4_2} = 1a$$

$$R_{3_2 5_2} = \frac{\sqrt{2\left(3+\sqrt{5}\right)}}{2}a$$

(6) For $3_1 4_2 5_1$

The vertex is of the form $n1_1 n2_1 n3_2$. Then

$$A_1 + A_2 + 2A_3 = \pi ----------(8)$$

$$\frac{\cos\frac{\pi}{n_1}}{\sin A_1} = \frac{\cos\frac{\pi}{n_2}}{\sin A_2} = \frac{\cos\frac{\pi}{n_3}}{\sin A_3} = k ---------(9)$$

$$(8) \Rightarrow \sin(A_1 + A_2) = \sin(\pi - 2A_3) = \sin 2A_3$$

$$\Rightarrow \sin A_1 \cos A_2 + \cos A_1 \sin A_2 = 2\sin A_3 \cos A_3$$

$$\Rightarrow \frac{1}{k}\cos\frac{\pi}{n_1}\cos A_2 + \cos A_1 \frac{1}{k}\cos\frac{\pi}{n_2} = 2\frac{1}{k}\cos\frac{\pi}{n_3}\cos A_3$$

$$\Rightarrow \left(2\cos\frac{\pi}{n_3}\cos A_3\right)^2 = \left(\cos\frac{\pi}{n_1}\cos A_2 + \cos A_1 \cos\frac{\pi}{n_2}\right)^2$$

$$\Rightarrow 4\cos^2\frac{\pi}{n_2}\cos^2 A_3 - \cos^2\frac{\pi}{n_1}\cos^2 A_2 - \cos^2\frac{\pi}{n_2}\cos^2 A_1$$

$$= 2\cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2}\cos A_1 \cos A_2$$

$$= 2\cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2}\left(\cos(A_1 + A_2) + \sin A_1 \sin A_2\right)$$

$$= 2\cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2}\left(-\cos 2A_3 + \sin A_1 \sin A_2\right)$$

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$\Rightarrow$

$$4\cos^2\frac{\pi}{n_3}\left(1 - \sin^2 A_3\right) - \cos^2\frac{\pi}{n_1}\left(1 - \sin^2 A_3 \frac{\cos^2\frac{\pi}{n_2}}{\cos^2\frac{\pi}{n_3}}\right) - \cos^2\frac{\pi}{n_2}\left(1 - \sin^2 A_3 \frac{\cos^2\frac{\pi}{n_1}}{\cos^2\frac{\pi}{n_3}}\right)$$

$$= 2\cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2}\left(2\sin^2 A_3 - 1 + \sin A_3 \frac{\cos\frac{\pi}{n_1}}{\cos\frac{\pi}{n_3}} \sin A_3 \frac{\cos\frac{\pi}{n_2}}{\cos\frac{\pi}{n_3}}\right)$$

$$\Rightarrow 4\sin^2 A_3\left(\cos^2\frac{\pi}{n_3} + \cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2}\right)$$

$$= 4\cos^2\frac{\pi}{n_3} - \cos^2\frac{\pi}{n_1} + 2\cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2} - \cos^2\frac{\pi}{n_2} = 4\cos^2\frac{\pi}{n_3} - \left(\cos\frac{\pi}{n_1} - \cos\frac{\pi}{n_2}\right)^2$$

$$\Rightarrow \sin A_3 = \frac{1}{2}\sqrt{\frac{4\cos^2\frac{\pi}{n_3} - \left(\cos\frac{\pi}{n_1} - \cos\frac{\pi}{n_2}\right)^2}{\cos^2\frac{\pi}{n_3} + \cos\frac{\pi}{n_1}\cos\frac{\pi}{n_2}}}$$

The calculated exact escribed radius value is

$$R_{3_1 4_2 5_1} = \frac{\sqrt{11+4\sqrt{5}}}{2} a$$

(7) For $3_1 4_4, 3_1 5_4$

The vertex is of the form $n1_1 n2_4$. Then

$$A_1 + 4A_2 = \pi \text{ ----------- } (8)$$

$$\frac{\cos\frac{\pi}{n_1}}{\sin A_1} = \frac{\cos\frac{\pi}{n_2}}{\sin A_2} = k \text{ --------- } (9)$$

$$(8) \Rightarrow \sin A_1 = \sin(\pi - 4A_2) = \sin 4A_2 = 2\sin 2A_2 \cos 2A_2 = 4\sin A_2 \cos A_2 \left(2\cos^2 A_2 - 1\right)$$

$$\Rightarrow \frac{1}{k}\cos\frac{\pi}{n_1} = 4\frac{1}{k}\cos\frac{\pi}{n_2}\left(2\cos^3 A_2 - \cos A_2\right)$$

$$\Rightarrow \cos^3 A_3 - \frac{1}{2}\cos A_3 - \frac{\cos\frac{\pi}{n_1}}{8\cos\frac{\pi}{n_2}} = 0$$

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This cubic equation must be solved to find the radius

(see Appendix A)

The calculated exact escribed radius values are

$$R_{3_1 4_1} = \frac{1}{2}\sqrt{\frac{24 - \left(\sqrt[3]{2\left(3\sqrt{3}+\sqrt{11}\right)} + \sqrt[3]{2\left(3\sqrt{3}-\sqrt{11}\right)}\right)^2}{18 - \left(\sqrt[3]{2\left(3\sqrt{3}+\sqrt{11}\right)} + \sqrt[3]{2\left(3\sqrt{3}-\sqrt{11}\right)}\right)^2}} a$$

$$R_{3_1 5_1} = \frac{1}{2}\sqrt{\frac{48 - \left(\sqrt[3]{6\left(\sqrt{3}+\sqrt{15}\right) + 2\sqrt{2\left(17+27\sqrt{5}\right)}} + \sqrt[3]{6\left(\sqrt{3}+\sqrt{15}\right) - 2\sqrt{2\left(17+27\sqrt{5}\right)}}\right)^2}{36 - \left(\sqrt[3]{6\left(\sqrt{3}+\sqrt{15}\right) + 2\sqrt{2\left(17+27\sqrt{5}\right)}} + \sqrt[3]{6\left(\sqrt{3}+\sqrt{15}\right) - 2\sqrt{2\left(17+27\sqrt{5}\right)}}\right)^2}} a$$

CHAPTER 3

OTHER DERIVATIONS

INTRODUCTION

Having found expressions for exact radii values of the Face and Vertex Regular Polyhedra any other geometrical property can be found. Here such formulae are stated and the geometrical properties calculated form such equations are given.

It is proven that sphere can be regarded as a regular polyhedron.

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3.1 SPHERE AS A LIMITING CASE OF A POLYHEDRON

Consider a regular polyhedron. There is only one type of polygon and hence $n_i = n$ and the number of polygons meet at a vertex is $M_i = M$.

The number of faces is given by

$$F = \frac{\frac{2}{n}}{\frac{1}{M}+\frac{1}{n}-\frac{1}{2}}$$

With $a =$ length of an edge of a polygon ,the radius of the escribed sphere is given by

$$R = \frac{a}{2}\frac{1}{\sqrt{1-\left(\frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{M}}\right)^2}} = \frac{\sin\frac{\pi}{M}}{\sqrt{\sin^2\frac{\pi}{M}-\cos^2\frac{\pi}{n}}} = \frac{\sin\frac{\pi}{M}}{\sqrt{\frac{1}{2}\left(\left(1-\cos\frac{2\pi}{M}\right)-\left(1+\cos\frac{2\pi}{n}\right)\right)}}$$

$$= \frac{\sin\frac{\pi}{M}}{\sqrt{-\cos\pi\left(\frac{1}{M}+\frac{1}{n}\right)\cos\pi\left(\frac{1}{M}-\frac{1}{n}\right)}} = \frac{\sin\frac{\pi}{M}}{\sqrt{\sin\pi\left(\frac{1}{M}+\frac{1}{n}-\frac{1}{2}\right)\cos\pi\left(\frac{1}{M}-\frac{1}{n}\right)}}$$

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Let a new variable define by $\alpha = \pi\left(\frac{1}{M}+\frac{1}{n}-\frac{1}{2}\right) \Rightarrow \frac{1}{M} = \frac{\alpha}{\pi}-\frac{1}{n}+\frac{1}{2}$

We can re write the expressions in terms of α as

$$F = \frac{\frac{2}{n}}{\frac{1}{M}+\frac{1}{n}-\frac{1}{2}} = \frac{\frac{2}{n}}{\frac{\alpha}{\pi}} = \frac{2\pi}{n\alpha} \quad \text{and}$$

$$R = \frac{\sin\frac{\pi}{M}}{\sqrt{\sin\pi\left(\frac{1}{M}+\frac{1}{n}-\frac{1}{2}\right)\cos\pi\left(\frac{1}{M}-\frac{1}{n}\right)}} = \frac{\sin\frac{\pi}{M}}{\sqrt{\sin\alpha\cos\pi\left(\frac{1}{M}-\frac{1}{n}\right)}}$$

$$= \frac{\sin\pi\left(\frac{\alpha}{\pi}-\frac{1}{n}+\frac{1}{2}\right)}{\sqrt{\sin\alpha\cos\pi\left(\frac{\alpha}{\pi}-\frac{1}{n}+\frac{1}{2}-\frac{1}{n}\right)}} = \frac{\cos\left(\alpha-\frac{\pi}{n}\right)}{\sqrt{\sin\alpha\sin\left(\frac{2\pi}{n}-\alpha\right)}}$$

With area of a polygon $= \dfrac{na^2}{4}\cot\dfrac{\pi}{n}$, the total surface area of the polyhedron , A is

A

= area of a polygon × total number of faces

= area of a polygon × F

$$= \left(\frac{na^2}{4}\cot\frac{\pi}{n}\right)\left(\frac{2\pi}{n\alpha}\right) = \frac{\pi a^2 \cot\dfrac{\pi}{n}}{2\alpha}$$

It is clear that both A and $R \to \infty$ as $\alpha \to 0$. But the limit

$$\lim_{\alpha\to 0}\frac{\text{total surface area}}{(\text{escribes sphere radius})^2} = \frac{\infty}{\infty}$$

$$= \lim_{\alpha\to 0}\frac{A}{R^2}$$

$$= \lim_{\alpha\to 0}\frac{\dfrac{\pi a^2\cot\dfrac{\pi}{n}}{2\alpha}}{\dfrac{a^2}{4}\,\dfrac{\cos^2\left(\alpha-\dfrac{\pi}{n}\right)}{\sin\alpha\sin\left(\dfrac{2\pi}{n}-\alpha\right)}} = 2\pi\cot\frac{\pi}{n}\lim_{\alpha\to 0}\frac{\sin\left(\dfrac{2\pi}{n}-\alpha\right)}{\cos^2\left(\alpha-\dfrac{\pi}{n}\right)}\,\frac{\sin\alpha}{\alpha} = 2\pi\frac{\cos\dfrac{\pi}{n}}{\sin\dfrac{\pi}{n}}\,\frac{\sin\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}}\,1 = 4\pi$$

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The angle α introduced here has a physical meaning as follows.

gap angle

= 2π - total angle provided by polygons at a vertex

= 2π - number of polygons × vertex angle of a polygon

$$= 2\pi - M \times \pi\left(\frac{n-2}{n}\right) = 2\pi M\left(\frac{1}{M}+\frac{1}{n}-\frac{1}{2}\right) = 2M\alpha$$

It is clear that when gap angle or $\alpha \to 0$, the polyhedron becomes a 2D tessellation.

When the length of a side of a polygon(a) is not $\to 0$ its surface is a plane(tessellation) which is having ∞ radius. But when the length of a side of a polygon $\to 0$ a sphere with a finite radius may be obtained.

This is conformed by the fact that we get 4π for the above ratio which is same as that for a sphere. Note that we have never used the equation for the surface area of the sphere in any of the derivations(see Appendix B).

This implies that the sphere can also be regarded as a limiting case of a polyhedron with its surface being a regular 2D tessellation.

It can be shown that the tessellation need not be regular.

3.2 OTHER FORMULAE

Following relations for the number of faces(F), vertices(V) and edges(E) are easily found by the Euler's formula $F + V = 2 + E$

(1) Number of faces [APPENDIX E]

$$F = \frac{2\sum_i \frac{M_i}{n_i}}{1 + \sum_i M_i \left(\frac{1}{n_i} - \frac{1}{2} \right)}$$

(2) Number of vertices

$$V = \frac{2}{1 + \sum_i M_i \left(\frac{1}{n_i} - \frac{1}{2} \right)}$$

(3) Number of edges

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$$E = \frac{\sum_i M_i}{1 + \sum_i M_i \left(\frac{1}{n_i} - \frac{1}{2} \right)}$$

Once the radius of the escribed sphere is found any other geometrical property of the polyhedra can be easily calculated. For example volume can be found considering pyramids formed by joining faces to the center of the escribed sphere. following are formulae for some properties.

(4) Angle subtended at the center by an edge(angle of polyhedron)

$$\theta = 2\cos^{-1}\left(\frac{\cos \frac{\pi}{n_i}}{\sin A_i} \right)$$

(5) Dihedral angle between adjacent faces[APPENDIX D]

$$\alpha_{edge} = \sum_{i,edge} \tan^{-1}\left(\frac{\cos A_i}{\sqrt{\sin^2 A_i - \cos^2 \frac{\pi}{n_i}}} \right)$$

(6) Internal solid angle of a vertex.[APPENDIX C]

$$\omega = 2\pi - \pi \sum_i M_i \pi + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin \frac{\pi}{n_i}} \right)$$

(7) Sum of total internal solid angles of vertices

$$\omega_{total} = 4\pi - 2V \sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin \frac{\pi}{n_i}} \right) - 2\pi F$$

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The calculated properties of the face and vertex regular polyhedra using these data are given under NUMERICAL DATA

Due to the fact that the regular polyhedra have only one type of polygons closed expressions can be obtained for their properties

(8) Number of edges $\quad E = \dfrac{2Mn}{2(M+n) - Mn}$

(9) Number of faces $\quad F = \dfrac{4M}{2(M+n) - Mn}$

(10) Number of vertices $\quad V = \dfrac{4n}{2(M+n) - Mn}$

(11) Radius of the escribed sphere

$$R = \frac{a}{2}\frac{\sin\frac{\pi}{M}}{\sqrt{\sin^2\frac{\pi}{M} - \cos^2\frac{\pi}{n}}}$$

(12) Radius of the inscribed sphere

$$r = \frac{a}{2}\frac{\cot\frac{\pi}{n}\cos\frac{\pi}{M}}{\sqrt{\sin^2\frac{\pi}{M} - \cos^2\frac{\pi}{n}}}$$

(13) Volume

$$v = \frac{Mna^3}{6(2(M+n)-Mn)}\frac{\cot^2\frac{\pi}{n}\cos\frac{\pi}{M}}{\sqrt{\sin^2\frac{\pi}{M} - \cos^2\frac{\pi}{n}}}$$

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(14) Angle subtended at the centre by an edge (angle of polyhedron)

$$\theta = 2\cos^{-1}\left(\frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{M}}\right)$$

3.3 NUMERICAL DATA

3.3.1 GENERAL DATA (ORDERED BY THE SOLID ANGLE OF A VERTEX)

NO	SYMBOL	NAME	NUMBER OF POLYGONS MEET AT A VERTEX	NUMBER OF FACES	NUMBER OF EDGES	NUMBER OF VERTICES	SUM OF VERTEX ANGES OF POLYGONS MEET AT A VERTEX/DEGREES
1	3_3	Tetrahedron	3	4	6	4	180
2	$4_2 3_1$	Triangular Prism	3	5	9	6	240
3	3_4	Octahedron	4	8	12	6	240
4	4_3	Hexahedron(Cube)	3	6	12	8	270
5	$3_3 4_1$	Square Anti Prism	4	10	16	8	270
6	$4_2 5_1$	Pentagonal Prism	3	7	15	10	288
7	$3_1 6_2$	Truncated Tetrahedron	3	8	18	12	300
8	$3_3 5_1$	Pentagonal Anti Prism	4	12	20	10	288
9	$4_2 6_1$	Hexagonal Prism	3	8	18	12	300
10	$3_3 6_1$	Hexagonal Anti Prism	4	14	24	12	300

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NO	SYMBOL	NAME	NUMBER OF POLYGONS MEET AT A VERTEX	NUMBER OF FACES	NUMBER OF EDGES	NUMBER OF VERTICES	SUM OF VERTEX ANGES OF POLYGONS MEET AT A VERTEX/DEGREES
11	$4_2 8_1$	Octagonal Prism	3	10	24	16	315
12	$3_2 4_2$	Cuboctahedron	4	14	24	12	300
13	$3_3 8_1$	Octagonal Anti Prism	4	18	32	16	315
14	$4_2 10_1$	Decagonal Prism	3	12	30	20	324
15	$3_3 10_1$	Decagonal Anti Prism	4	22	40	20	324
16	$4_2 12_1$	Dodecagonal Prism	3	14	36	24	330
17	3_5	Icosahedron	5	20	30	12	300
18	$3_3 12_1$	Dodecagonal Anti Prism	4	26	48	24	330
19	$3_1 8_2$	Truncated Cube	3	14	36	24	330
20	5_3	Dodecahedron	3	12	30	20	324

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NO	SYMBOL	NAME	NUMBER OF POLYGONS MEET AT A VERTEX	NUMBER OF FACES	NUMBER OF EDGES	NUMBER OF VERTICES	SUM OF VERTEX ANGES OF POLYGONS MEET AT A VERTEX/DEGREES
21	$4_1 6_2$	Truncated Octahedron	3	14	36	24	330
22	$3_1 4_3$	Small Rhombicuboctahedron	4	26	48	24	330
23	$3_4 4_1$	Snub Cube	5	38	60	24	330
24	$3_2 5_2$	Icosidodecahedron	4	32	60	30	336
25	$3_1 10_2$	Truncated Dodecahedron	3	32	90	60	348
26	$4_1 6_1 8_1$	Great Rhombicuboctahedron	3	26	72	48	345
27	$5_1 6_2$	Truncated Icosahedron	3	32	90	60	348
28	$3_1 4_2 5_1$	Small Rhombicosidodecahedron	4	62	120	60	348
29	$3_4 5_1$	Snub Dodecahedron	5	92	150	60	348
30	$4_1 6_1 10_1$	Great Rhombicosidodecahedron	3	62	180	120	354

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Table 3.1

3.3.2 DATA CALCULATED FROM DERIVED EQUATIONS (ORDERED BY THE SOLID ANGLE OF A VERTEX)

NO	SYMBOL	NAME	SOLID ANGLE OF A VERTEX/4PI sr	RADIUS /LENGTH OF AN EDGE	VOLUME/ LENGTH OF AN EDGE^3	VOLUME/ RADIUS^3	ANGLE SUBTENDED AT THE CENTRE BY AN EDGE/ DEGREES
1	3_3	Tetrahedron	0.04386991402295545262 8	0.61237243569579452455	0.12	0.5132	109.47
2	$4_2 3_1$	Triangular Prism	0.08333333333333333333	0.76376261582597333443	0.43	0.9719	81.79
3	3_4	Octahedron	0.10817344796939272983	0.70710678118654752440	0.47	1.3333	90.00
4	4_3	Hexahedron(Cube)	0.12500000000000000000	0.86602540378443864676	1.00	1.5396	70.53
5	$3_3 4_1$	Square Anti Prism	0.14274378718068905088	0.82266438800803628873	0.96	1.7189	74.86
6	$4_2 5_1$	Pentagonal Prism	0.15000000000000000000	0.98671515532598310732	1.72	1.7909	60.89
7	$3_1 6_2$	Truncated Tetrahedron	0.15204336199234818246	1.17260393995515738 86	2.71	1.6812	50.48
8	$3_3 5_1$	Pentagonal Anti Prism	0.16389445018831418952	0.95105651629515357212	1.58	1.8352	63.43
9	$4_2 6_1$	Hexagonal Prism	0.16666666666666666667	1.11803398874989484 82	2.60	1.8590	53.13
10	$3_3 6_1$	Hexagonal Anti Prism	0.17811477836587375037	1.08766387358053743 69	2.34	1.8167	54.74

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NO	SYMBOL	NAME	SOLID ANGLE OF A VERTEX/4PI sr	RADIUS /LENGTH OF AN EDGE	VOLUME/ LENGTH OF AN EDGE^3	VOLUME/ RADIUS^3	ANGLE SUBTENDED AT THE CENTRE BY AN EDGE/ DEGREES
11	$4_2 8_1$	Octagonal Prism	0.18750000000000000000	1.3989663259659067020	4.83	1.7635	41.88
12	$3_2 4_2$	Cuboctahedron	0.19591327601530363509	1.0000000000000000000	2.36	2.3570	60.00
13	$3_3 8_1$	Octagonal Anti Prism	0.19599139196000959929	1.3755485807735077127	4.27	1.6398	42.63
14	$4_2 10_1$	Decagonal Prism	0.20000000000000000000	1.6935270853310539386	7.69	1.5841	34.34
15	$3_3 10_1$	Decagonal Anti Prism	0.20675875319410803684	1.6745047437425603068	6.75	1.4375	34.75
16	$4_2 12_1$	Dodecagonal Prism	0.20833333333333333333	1.9955076566049245038	11.20	1.4090	29.02
17	3_5	Icosahedron	0.20965059100153751343	0.95105651629515357212	2.18	2.5362	63.43
18	$3_3 12_1$	Dodecagonal Anti Prism	0.21395022502107160677	1.9795119433363656367	9.78	1.2611	29.26
19	$3_1 8_2$	Truncated Cube	0.22295663800765181754	1.7788236456639244509	13.60	2.4162	32.65
20	5_3	Dodecahedron	0.23568771323782495563	1.4012585384440735447	7.66	2.7852	41.81

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NO	SYMBOL	NAME	SOLID ANGLE OF A VERTEX/4PI sr	RADIUS /LENGTH OF AN EDGE	VOLUME/ LENGTH OF AN EDGE^3	VOLUME/ RADIUS^3	ANGLE SUBTENDED AT THE CENTRE BY AN EDGE/ DEGREES
21	$4_1 6_2$	Truncated Octahedron	0.25000000000000000000	1.58113883008418966600	11.31	2.8622	36.87
22	$3_1 4_3$	Small Rhombicuboctahedron	0.27704336199234818246	1.39896632596590670200	8.71	3.1827	41.88
23	$3_4 4_1$	Snub Cube	0.27565364345454073491	1.34371337374460170130	7.89	3.2518	43.69
24	$3_2 5_2$	Icosidodecahedron	0.29234795477416835754	1.61803398874989484820	13.84	3.2661	36.00
25	$3_1 10_2$	Truncated Dodecahedron	0.30806988179969249731	2.96944901586339846700	85.04	3.2478	19.39
26	$4_1 6_1 8_1$	Great Rhombicuboctahedron	0.31250000000000000000	2.31761091289276651380	41.80	3.3577	24.92
27	$5_1 6_2$	Truncated Icosahedron	0.33810409558739168146	2.47801865906761553760	55.29	3.6334	23.28
28	$3_1 4_2 5_1$	Small Rhombicosidodecahedron	0.35382602261291582123	2.23295050941569004950	41.62	3.7378	25.88
29	$3_4 5_1$	Snub Dodecahedron	0.35886935933301325883	2.15583737511563970180	37.62	3.7543	26.82
30	$4_1 6_1 10_1$	Great Rhombicosidodecahedron	0.37500000000000000000	3.80239449985129358480	206.80	3.7617	15.11

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Table 3.2

CHAPTER 4

APPLICATION TO FINITE ELEMENT ANALYSIS

INTRODUCTION

The relation between the finite element analysis and tessellations lies on the fact that tessellations can cover 2D or 3D space. Here possibility of using these tessellations in finite element analysis is analyzed.

Finite element analysis requires a region to be divided into non overlapping sub regions called finite elements. A method for dividing the region into finite elements and a method for defining the Lagrange interpolating polynomial are investigated. With the piecewise polynomial selected in the above manner the limitations of the regular tessellations as finite elements are investigated.

4.1 FINITE DIFFERENCE AND FINITE ELEMENT METHODS

There are two main numerical techniques to solve partial differential equations. They are the finite difference method and the finite element method.

In both methods the region of the range of the problem is discretized in to non overlapping sub regions which are called finite elements. Hence these methods have a strong connection with geometry.

In finite difference method the region is discretized in to finite elements with their sides parallel to variable axes. In contrast to this in the finite element method the region is discretized in to finite elements in any suitable way and the function of concern is assumed going over and through points above these regions(interpolating function is found). Because of the flexibility of the choose of the regions, finite element method is preferred over the finite difference method. Here we restrict the discussion to finite element method.

The ideas discussed are not restricted to the solution of partial differential equations. They are equally applicable to numerical differentiation, numerical integration etc.

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4.2 FINITE ELEMENT METHODS [5]

Let partial the differential equation be written in operator form as $L(V) = r$ within the region R.To apply the finite element methods we divide the region in to non overlapping finite elements e. We approximate the original function within each finite element $R^{(e)}$ as follows.

$$V^{(e)} = N^{(e)} v^{(e)} = \overline{N}^{(e)} v$$

Where $N^{(e)}$ are shape functions and $v^{(e)}$ are the values of the original function. Here $\overline{N}^{(e)}$ is the extended shape function to include all the function values v within the region $R^{(e)}$.

We can write the total function for the region R as

$$V = \sum_{e=1}^{M} V^{(e)} = \sum_{e=1}^{M} \overline{N}^{(e)} v = \left(\sum_{e=1}^{M} \overline{N}^{(e)} \right) v = \overline{N} v$$

The shape functions are found as follows.

Normally the function is assumed as a Lagrange polynomial and written as $V(X) = f(X)A$ where $f(X)$ is the polynomial terms and A is the set of coefficients to be determined. Here $X = (x, y)$ in 2D $X = (x, y, z)$ in 3D.

Normally the function values are set at nodes of the finite element and hence the node set of the finite element is $P = (X)$. To find the shape functions we evaluate the function at each node and find the coefficients A as follows.

$$V(P) = v = f(P)A = BA$$
$$V(X) = f(X)B^{-1}v$$
$$V(X) = Nv$$

To find the Lagrange polynomial $V(X)$ we need to find the shape matrix $N = f(X)(f(P))^{-1}$.

To find the shape matrix we need to find B^{-1}. Hence B should be non singular.

The matrix $B = f(P)$ is only depend on the geometry of the finite element in the form of vertex set P and the selected piecewise polynomial f.

Weighted residual and Variational methods are the main methods of solving differential equations by finite elements. All these methods are based on some integral and the integral over the region is the sum of element contributions. Hence we can substitute the assumed polynomial $V^{(e)}(X)$ in the element integral and come up with the total integral.

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4.3 WEIGHTED RESIDUAL METHODS

In these methods the residual due to the substitution of the piecewise polynomial to the differential equation is found. Its weighted integral is used to find the function values.

$$E(V) = L(V) - r$$
$$E^{(e)}(V^{(e)}) = L(V^{(e)}) - r$$

4.3.1 LEAST SQUARE METHOD

This is a main weighted residual method where the weight is taken to be E itself.

$$WE(V) = \int_R E^2 dR$$

$$\frac{\partial WE}{\partial v_i} = 2\int_R E\frac{\partial E}{\partial v_i}dR = \sum_{e=1}^{M} 2\int_{R^{(e)}} E^{(e)}\frac{\partial E^{(e)}}{\partial v_i}dR^{(e)} = 0$$

$$\frac{\partial WE^{(e)}}{\partial v^{(e)}} = 2\int_{R^{(e)}} E^{(e)}\frac{\partial E^{(e)}}{\partial v^{(e)}}dR^{(e)} = \underline{0}$$

4.3.2 GALERKIN METHOD

This is a main weighted residual method where the weight is taken to be $N^{(e)}$.

$$\int_R \overline{N}E(V)dR = \int_R\left(\sum_{e=1}^{M}\overline{N}^{(e)}\right)E(V)dR = \sum_{e=1}^{M}\left(\int_{R^{(e)}}\overline{N}^{(e)}E^{(e)}(V^{(e)})dR^{(e)}\right) = \int_{R^{(e)}}\overline{N}^{(e)}E^{(e)}(V^{(e)})dR^{(e)} = \underline{0}$$

4.4 VARIATIONAL METHODS

This is a method based on the criterion of the calculus of variations.

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4.4.1 RITZ METHOD

In this method the given differential equation is written as a the Euler equation of some variational problem.

$$J(V) = \int_R F(V)dR = \sum_{e=1}^{M}\int_{R^{(e)}}F^{(e)}(V^{(e)})dR^{(e)} = \sum_{e=1}^{M}J^{(e)} \Rightarrow L(V) = r$$

$$\frac{\partial J(V)}{\partial v} = \begin{bmatrix}\frac{\partial J}{\partial v_1} & \frac{\partial J}{\partial v_2} & & \frac{\partial J}{\partial v_N}\end{bmatrix}^T = \underline{0}$$

$$\frac{\partial J}{\partial v_i} = \sum_{e=1}^{M}\frac{\partial J^{(e)}}{\partial v_i} = 0$$

4.5 USE OF REGULAR TESSELLATIONS IN FINITE ELEMENTS

As discussed earlier tessellations do cover infinite regions and can be made to cover a finite region of arbitrary shape if the size of elements are made small according to the accuracy requirement. This is the link with tessellations and finite elements. As we did earlier we restrict overselves to 2D tessellations made with regular polygons and 3D tessellations made with face and vertex regular polyhedra.

Criteria

1) Interior as a regular tessellation and
2) Boundary by different elements.

Or

1) Whole region as a regular tessellation
2) Boundary achieved by making the size of the elements small.

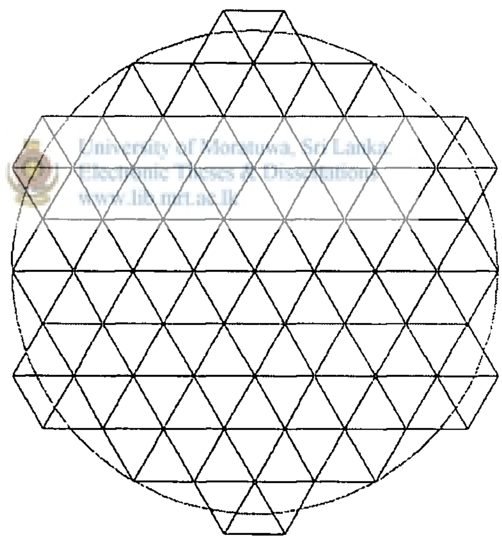
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Figure 4.1

Advantages

1) Easy discretitation of the region in to finite elements

(Regular polygons have a escribed circle and face and vertex regular polyhedra have a escribed sphere. So this is a matter of filling the region by overlapping circles or spheres).

2) Easy computation and interpretation

(Since properties of the finite elements used are known. Each node is situated at a constant distance away from the neighboring nodes).

3) Higher degree of accuracy

(if we select polygons or polyhedra with higher number of nodes as finite elements and/or if we decide to make the size of finite elements small to achieve the boundary).

4.6 CHOOSING LAGRANGE POLYNOMIAL OF MORE THAN ONE VARIABLE.

We also need to choose the two variable(or higher) Lagrange polynomial $V(X)$ for given number of points. Unlike in one variable polynomial which has only one term for one degree there is no unique polynomial in two variables since there is more than one term corresponding to one degree.

I propose the following criteria of selecting the polynomial

Criteria

1. Select the complete polynomial of immediate lesser number of terms.
2. Select the other terms from the immediate symmetric higher degree terms.
3. When there is more than one possibility always select terms with more types of product terms.

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Advantage of each procedure is

1. Complexity of calculation due to higher degree terms is avoided.
2. Allow the function to take any arbitrary value irrespective of the point.
3. Allow the function to vary arbitrarily in both positive and negative directions.

4.7 FINITE ELEMENT ANALYSIS IN 2D

4.7.1 POSSIBLE REGULAR 2D TESSELLATIONS

Vertex angle of a regular polygon of n number of sides is given by

$$\pi - \frac{2\pi}{n} = \pi\left(1 - \frac{2}{n}\right).$$

To construct a 2D tessellation we require that the sum of vertex angles is 2π which is the sum of plane angles around a point. If M number of polygons used at a vertex this relation reeds as

$$\pi\left(1 - \frac{2}{n}\right)M = 2\pi \text{ or } \frac{1}{n} + \frac{1}{M} = \frac{1}{2}$$

It is easily seen that $3 \leq M \leq 6$. So we left with only a finite number of solutions for (n, M) which is symbolized as n_M given by $3_6, 4_4, 6_3$. This means that no more than Equilateral Triangle(3), Square(4), Regular Hexagon(6) will cover 2D space.
The corresponding regular 2D tessellations are given below.

(1) 3_6

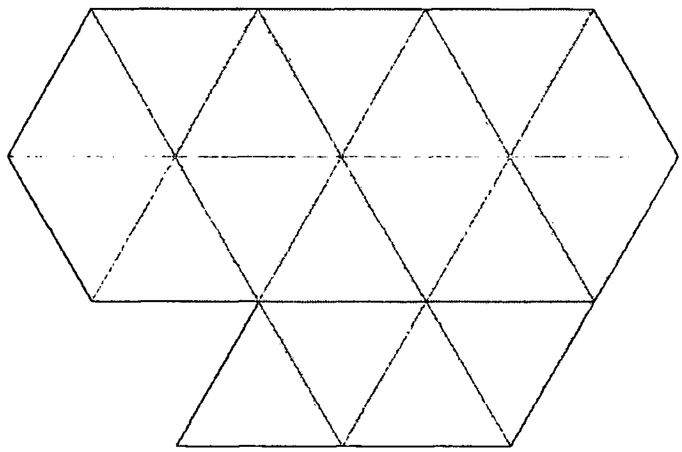

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Figure 4.2

(2) 4_4

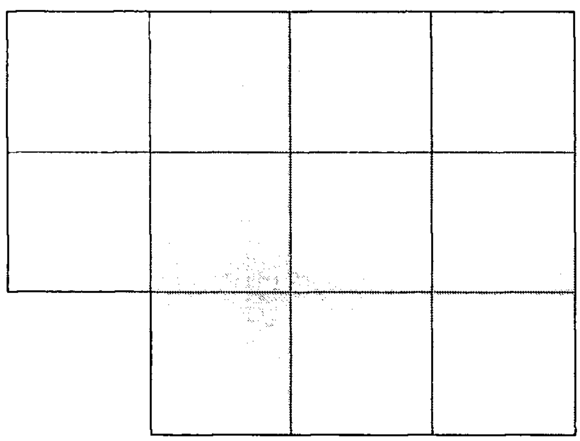

Figure 4.3

(3) 6_3

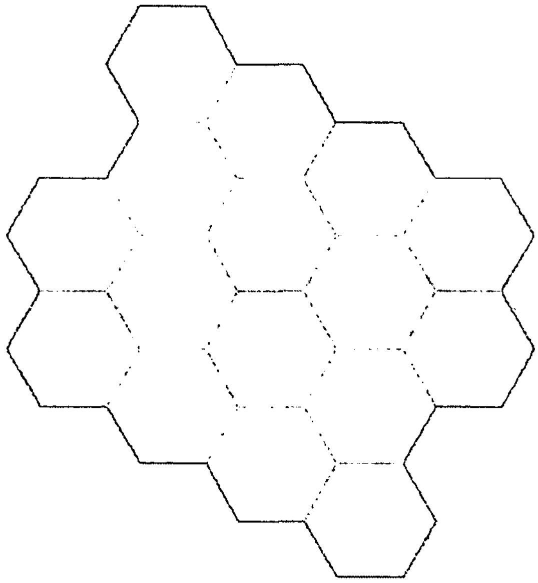

Figure 4.4

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4.7.2 2D TESSELLATIONS IN FINITE ELEMENTS

We have categorized all the possible kinds of 2D space filling or tessellations using Regular polygons. They were categorized as

1. Regular 2D Tessellations : 3 types(discussed).
2. Semi-Regular 2D Tessellations : 8 types.

Here we restrict ourselves to regular 2D tessellations only.

In 2D finite elements, it can be shown that the number of terms in 2 variable Lagrange polynomial is equal to

$$T = \sum_{r=1}^{N} {}^{2}H_{r} = \sum_{r=1}^{N} {}^{2+r-1}C_{r} = {}^{N+2}C_{2} = \frac{(N+2)(N+1)}{1.2}$$

The number and nature of terms are given in the following table

degree	terms correspond to	terms	partial sum	sum	cumulative sum
0	0	1	1	1	1
1	1	x, y	2	2	3
2	2	x^2, y^2	2	3	6
	1+1	xy	1		
3	3	x^3, y^3	2	4	10
	2+1	x^2y, y^2x	2		
4	4	x^4, y^4	2	5	15
	3+1	x^3y, y^3x	2		
	2+2	x^2y^2	1		

Table 4.1

4.7.3 REGULAR 2D TESSELLATIONS IN FINITE ELEMENTS

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(1) Equilateral Triangle(3).

The selected polynomial by the above criteria is $V(x, y) = a_1 + a_2x + a_3y$.

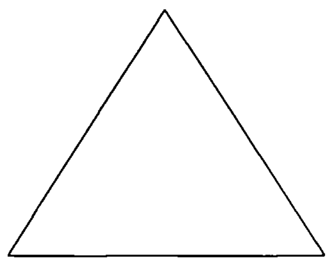

Figure 4.5

(2) Square(4).

The selected polynomial by the above criteria is $V(x, y) = a_1 + a_2x + a_3y + a_4xy$.

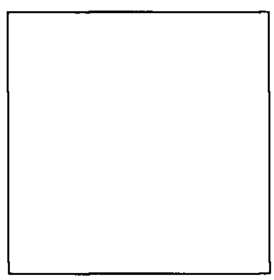

Figure 4.6

(3) Regular Hexagon(6).

The selected polynomial by the above criteria is

$V(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$.

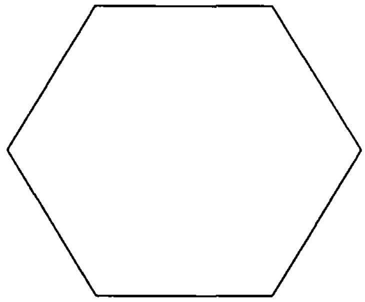

Figure 4.7

4.7.4 LIMITATIONS OF REGULAR POLYGONS AS FINITE ELEMENTS

(1) Equilateral Triangle(3).

The selected polynomial is $V(x,y) = a_1 + a_2 x + a_3 y$. For any other orientation we can transform the coordinates by $x = pX + qY + r$ and $y = uX + vY + w$. We obtain $V(X,Y) = A_1 + A_2 X + A_3 Y$ which is similar to the original equation. So both A_i and a_i exist or not exist together. Therefore all the orientations are such that either B is singular or non singular.

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Consider the following orientation with length of an edge is $2\sqrt{3}$ the coordinate set of nodes are $P = \{(0,2),(-1,-\sqrt{3}),(1,-\sqrt{3})\}$.

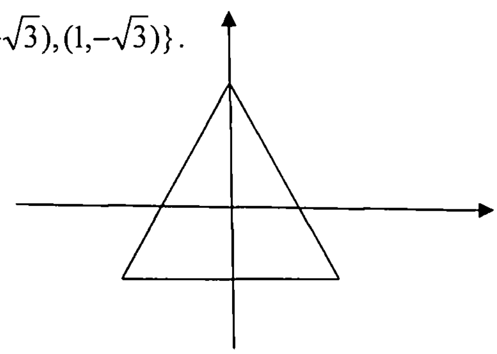

Figure 4.8

Here $|B| = \begin{vmatrix} 1 & 0 & 2 \\ 1 & -1 & -\sqrt{3} \\ 1 & 1 & -\sqrt{3} \end{vmatrix} = 4 + 2\sqrt{3} \neq 0$

Hence for any other orientation matrix B is non singular.

Therefore equilateral triangle can be used as a finite element in any orientation.

(2) Square(4).

The selected polynomial is $V(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$. For any other orientation we can transform the coordinates by $x = pX + qY + r$ and $y = uX + vY + w$. We obtain $V(X,Y) = A_1 + A_2 X + A_3 Y + A_4 X^2 + A_5 Y^2 + A_6 XY$ which is not the same as the original equation. Hence we can't predict the behavior of B using the above technique. It can be singular or non singular depending on the orientation.

Consider the following orientation with length of an edge is 2 the coordinate set of nodes are $P = \{(1,1),(1,-1),(-1,-1),(1,-1)\}$.

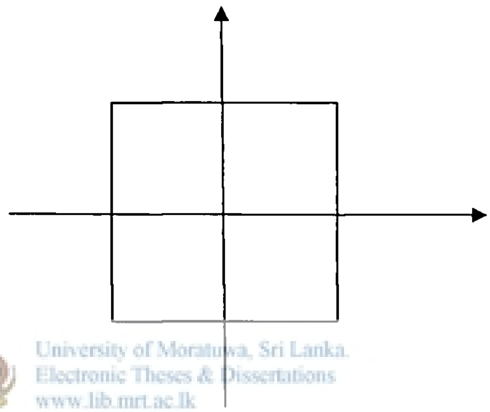

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Figure 4.9

Here $|B| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{vmatrix} = -16 \neq 0$

Therefore square can be used as a finite element in this orientation.

Square can be placed in such a way that all its nodes lie on the two axes as follows.

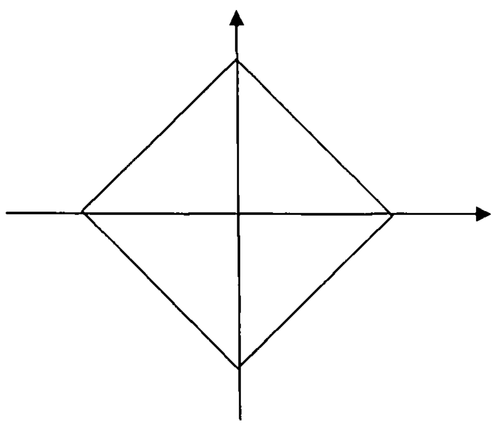

Figure 4.10

B is singular in this orientation. This is because all the nodes has at least one of x or y zero and the polynomial contains a xy term.

But there is only one orientation where this occurs with coordinate set of nodes are $P = \{(1,0),(0,1),(-1,0),(0,-1)\}$ if length of an edge is $\sqrt{2}$.

(3) Regular Hexagon(6).

The selected polynomial is $V(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$. For any other orientation we can transform the coordinates by $x = pX + qY + r$ and $y = uX + vY + w$. We obtain $V(X,Y) = A_1 + A_2X + A_3Y + A_4X^2 + A_5Y^2 + A_6XY$ which is same as the original equation. So both A_i and a_i exist or not exist together. Therefore all the orientations are such that either B is singular or non singular.

For the following orientation with length of an edge is 2 the coordinate set of nodes are $P = \{(2,0),(1,\sqrt{3}),(-1,\sqrt{3}),(-2,0),(-1,-\sqrt{3}),(1,-\sqrt{3})\}$

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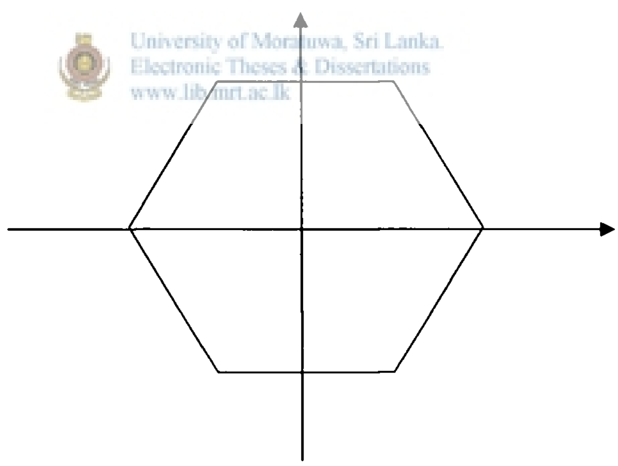

Figure 4.11

Here $B = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 & 0 \\ 1 & 1 & \sqrt{3} & 1 & \sqrt{3} & 3 \\ 1 & - & \sqrt{3} & 1 & -\sqrt{3} & 3 \\ 1 & -2 & 0 & 4 & 0 & 0 \\ 1 & -1 & -\sqrt{3} & 1 & \sqrt{3} & 3 \\ 1 & 1 & -\sqrt{3} & 1 & -\sqrt{3} & 3 \end{pmatrix}$

We need to find whether B is singular or not. For that we perform elementary raw operations as follows.

$$B=\begin{pmatrix}1&2&0&4&0&0\\1&1&\sqrt{3}&1&\sqrt{3}&3\\1&-1&\sqrt{3}&1&-\sqrt{3}&3\\1&-2&0&4&0&0\\1&-1&-\sqrt{3}&1&\sqrt{3}&3\\1&1&-\sqrt{3}&1&-\sqrt{3}&3\end{pmatrix}\begin{array}{l}\dfrac{-R_5+R_6}{2}\to R_6\\\dfrac{-R_4+R_1}{4}\to R_1\\\dfrac{-R_3+R_2}{2}\to R_2\end{array}$$

$$\sim\begin{pmatrix}0&1&0&0&0&0\\0&1&0&0&\sqrt{3}&0\\1&-1&\sqrt{3}&1&-\sqrt{3}&3\\1&-2&0&4&0&0\\1&-1&-\sqrt{3}&1&\sqrt{3}&3\\0&1&0&0&-\sqrt{3}&0\end{pmatrix}\frac{R_2+R_6}{2}\to R_6$$

$$\sim\begin{pmatrix}0&1&0&0&0&0\\0&1&0&0&\sqrt{3}&0\\1&-1&\sqrt{3}&1&-\sqrt{3}&3\\1&-2&0&4&0&0\\1&-1&-\sqrt{3}&1&\sqrt{3}&3\\0&1&0&0&0&0\end{pmatrix}-R_6+R_1\to R_1$$

$$\sim\begin{pmatrix}0&0&0&0&0&0\\0&1&0&0&\sqrt{3}&0\\1&-1&\sqrt{3}&1&-\sqrt{3}&3\\1&-2&0&4&0&0\\1&-1&-\sqrt{3}&1&\sqrt{3}&3\\0&1&0&0&0&0\end{pmatrix}$$

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Therefore matrix B is singular. (same thing may be shown by performing $4C_1-C_4-C_6\to C_1$)

Hence for any other orientation matrix B is singular.

Therefore regular hexagon cannot be used as a finite element in any orientation.

4.7.5 PROOF OF A GENERAL RESULT

1. Suppose that we have a regular polygon of n number of sides with unit escribed sphere radius. Then the coordinate set of vertices is $P=\left\{\left(\cos\frac{2\pi}{n}i,\sin\frac{2\pi}{n}i\right)\middle| i=1,2,....,n\right\}$.

2. Suppose we select the nodes(points where function values are assumed) at vertices.

3. Suppose we have the complete polynomial of degree 2 ($1, X, Y, X^2, XY, Y^2$ terms) in the Lagrange polynomial $V(X,Y) = f(X,Y)A$.

4. Since $1 = \cos^2\left(\frac{2\pi}{n}i\right) + \sin^2\left(\frac{2\pi}{n}i\right)$, the columns of $f(P)$ correspond to $1, X^2, Y^2$ are linearly dependent.

5. Hence $|f(P)| = 0$

6. Hence we can't use the above regular polygon as a finite element.

We will show here that this is independent of the coordinate axes.

1. Suppose that the above coordinate system is X, Y and any scaling, translation or rotation of the above coordinate system can be represented by the x, y coordinate system where $x = pX + qY + r$ and $y = uX + vY + w$.

2. Then the columns of $V(x,y) = V(pX + qY + r, uX + vY + w)$ corresponding to $1, x, y, x^2, xy, y^2$ will be 6 linear combinations of columns of $V(X,Y)$ corresponding to $1, X, Y, X^2, XY, Y^2$.

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3. Earlier we showed that $f(P)$ where $P = (X,Y)$ is singular or has dependent columns.

4. Now $f(P')$ where $P' = (x,y)$ has columns which are linear combinations of columns of $f(P)$ which are linearly dependent.

5. We have the theorem "if S is a set of n linearly dependent vectors than any set of n or higher number of vectors spanned by S are linearly dependent".

6. Hence $f(P')$ has linearly dependent columns.

7. Therefore $|f(P')| = 0$

8. Hence we can't use bi axis symmetric versions of the above polygons as finite elements in any orientation.

4.7.6 CONCLUSION

Any polygon having two axis of symmetry with nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains the complete polynomial of degree two.

4.7.7 DEDUCTIONS

(1) With the Lagrange polynomial selected in the above manner the only possible regular polygons that can be used as finite element in 2D are Equilateral Triangle and Square.

(2) Regular Hexagon cannot be used as a finite element since the piecewise polynomial is a 2D complete polynomial.

Therefore the only possible regular 2D tessellations in finite element analysis are 3_6 and 4_4.

The corresponding finite elements are **Equilateral Triangle** and **Square.**

(3) From the other tessellations only the following tessellations containing equilateral triangles and squares are possible in finite element analysis

1. $3_3 4_2$
2. $3_2 4_1 3_1 4_1$

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4.8 FINITE ELEMET ANALYSIS IN 3D

4.8.1 POSSIBLE REGULAR 3D TESSELLATIONS

A necessary condition for the existence of a 3D tessellation is that the sum of solid angles of the polyhedra meet at a vertex should be 4π.

The solid angle of a vertex of a regular polyhedron is given by

$$\omega = 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\dfrac{\pi}{n_i}}\right)$$

For the possible types of regular 3D tessellations we verify that this requirement is met. For all the possible types it will be found out that polyhedra are of the form $n1_1 n2_2$. Hence the angles A_1 and A_2 are found to be

$$\cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}}$$

and

$$\cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}} = \frac{1}{2}\frac{\cos\frac{\pi}{n_1}}{\cos\frac{\pi}{n_2}} = \frac{1}{2}\frac{\sin A_1}{\sin A_2}$$

$$\Rightarrow \sin A_1 = \sin 2A_2$$

$$\Rightarrow A_1 = 2A_2 \text{ or } A_1 = \pi - 2A_2$$

(1) $3_1 4_2$

Here $n_1 = 3$ and $n_2 = 4$

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$$\cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}} = \frac{\cos\frac{\pi}{3}}{2\cos\frac{\pi}{4}} = \frac{\frac{1}{2}}{2\left(\frac{1}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}}$$

And

$$\cos A_1 = \cos(\pi - 2A_2) = -2\cos^2 A_2 + 1 = -2\left(\frac{1}{2\sqrt{2}}\right)^2 + 1 = \frac{3}{4}$$

Then

$$\frac{\cos A_1}{\sin\frac{\pi}{n_1}} = \frac{\cos A_1}{\sin\frac{\pi}{3}} = \frac{\frac{3}{4}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \quad \text{and} \quad \frac{\cos A_2}{\sin\frac{\pi}{n_2}} = \frac{\cos A_2}{\sin\frac{\pi}{4}} = \frac{\frac{1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{1}{2} = \sin\frac{\pi}{6}$$

So $\omega = 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right) = 2\pi - \pi(1+2) + 2\left(1.\frac{\pi}{3} + 2.\frac{\pi}{6}\right) = \frac{\pi}{3}$

Therefore $12\omega = 4\pi$.

So if the 3D tessellation exists it should be of the form $(3_1 4_2)_{12}$ which actually exists.

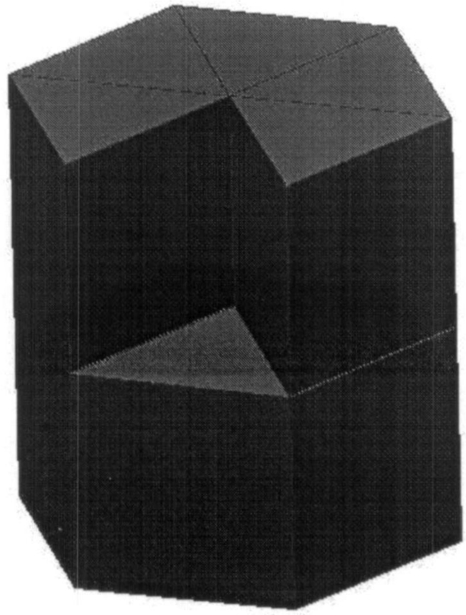

Figure 4.12

(1) $4_3 = 4_1 4_2$

Here $n_1 = 4$ and $n_2 = 4$

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$$\cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}} = \frac{\cos\frac{\pi}{4}}{2\cos\frac{\pi}{4}} = \frac{1}{2}$$

And

$$\cos A_1 = \cos(\pi - 2A_2) = -2\cos^2 A_2 + 1 = -2\left(\frac{1}{2}\right)^2 + 1 = \frac{1}{2}$$

Then

$$\frac{\cos A_1}{\sin\frac{\pi}{n_1}} = \frac{\cos A_1}{\sin\frac{\pi}{4}} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \quad \text{and} \quad \frac{\cos A_2}{\sin\frac{\pi}{n_2}} = \frac{\cos A_2}{\sin\frac{\pi}{4}} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

So $\omega = 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right) = 2\pi - \pi(1+2) + 2\left(1.\frac{\pi}{4} + 2.\frac{\pi}{4}\right) = \frac{\pi}{2}$

Alternatively we view this polyhedron as a regular one 4_3 for which

$n_i = 4$ and $M_i = 3$. Then to find A_i

$$\sum_i M_i A_i = M_i A_i = \pi \Rightarrow A_i = \frac{\pi}{M_i} = \frac{\pi}{3} \Rightarrow \frac{\cos A_i}{\sin\frac{\pi}{n_i}} = \frac{\cos\frac{\pi}{M_i}}{\sin\frac{\pi}{n_i}} = \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{4}} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\omega = 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right) = 2\pi - \pi(3) + 2.3.\frac{\pi}{4} = \frac{\pi}{2}$$

Therefore $8\omega = 4\pi$.

So if the 3D tessellation exists it should be of the form $(4_1 4_2)_8 = (4_3)_8$ which actually exists.

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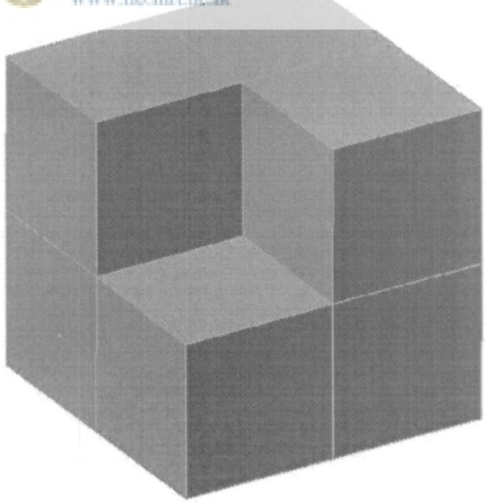

Figure 4.13

(1) $6_1 4_2$

Here $n_1 = 6$ and $n_2 = 4$.

$$\cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}} = \frac{\cos\frac{\pi}{6}}{2\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{3}}{2}}{2\left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{3}}{2\sqrt{2}}$$

And

$$\cos A_1 = \cos(\pi - 2A_2) = -2\cos^2 A_2 + 1 = -2\left(\frac{3}{2\sqrt{2}}\right)^2 + 1 = \frac{1}{4}$$

Then

$$\frac{\cos A_1}{\sin\frac{\pi}{n_1}} = \frac{\cos A_1}{\sin\frac{\pi}{6}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = \sin\frac{\pi}{6} \quad \text{and} \quad \frac{\cos A_2}{\sin\frac{\pi}{n_2}} = \frac{\cos A_2}{\sin\frac{\pi}{4}} = \frac{\frac{\sqrt{3}}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\text{So } \omega = 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right) = 2\pi - \pi(1+2) + 2\left(1.\frac{\pi}{6} + 2.\frac{\pi}{3}\right) = \frac{2\pi}{3}$$

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Therefore $6\omega = 4\pi$.

So if the 3D tessellation exists it should be of the form $(6_1 4_2)_6$ which actually exists.

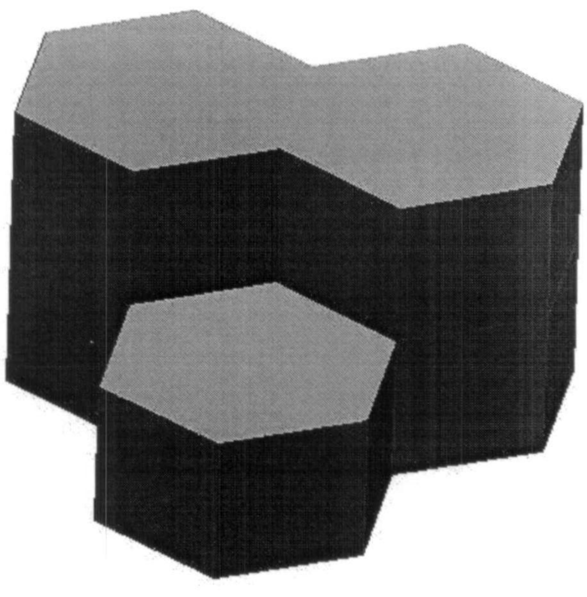

Figure 4.14

(1) $4_1 6_2$

Here $n_1 = 4$ and $n_2 = 6$

$$\cos A_2 = \frac{\cos\frac{\pi}{n_1}}{2\cos\frac{\pi}{n_2}} = \frac{\cos\frac{\pi}{4}}{2\cos\frac{\pi}{6}} = \frac{\frac{1}{\sqrt{2}}}{2\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{6}}$$

And

$$\cos A_1 = \cos(\pi - 2A_2) = -2\cos^2 A_2 + 1 = -2\left(\frac{1}{\sqrt{6}}\right)^2 + 1 = \frac{2}{3}$$

Then

$$\frac{\cos A_1}{\sin\frac{\pi}{n_1}} = \frac{\cos A_1}{\sin\frac{\pi}{4}} = \frac{\frac{2}{3}}{\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{3} \quad \text{and} \quad \frac{\cos A_2}{\sin\frac{\pi}{n_2}} = \frac{\cos A_2}{\sin\frac{\pi}{6}} = \frac{\frac{1}{\sqrt{6}}}{\frac{1}{2}} = \frac{2}{\sqrt{6}}$$

So

$$\omega = 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right)$$

$$= 2\pi - \pi(1+2) + 2\left(\sin^{-1}\frac{2\sqrt{2}}{3} + 2\sin^{-1}\frac{2}{\sqrt{6}}\right) = -\pi + 2\sin^{-1}\left(\frac{2\sqrt{2}}{3}.\left(1 - 2.\frac{4}{6}\right) + \frac{1}{3}.2.\frac{2}{\sqrt{6}}.\frac{\sqrt{2}}{\sqrt{6}}\right)$$

$$= -\pi + 2\sin^{-1} 0 = -\pi + 2\pi = \pi$$

Therefore $4\omega = 4\pi$.

So if the 3D tessellation exists it should be of the form $(4_1 6_2)_4$ which actually exists.

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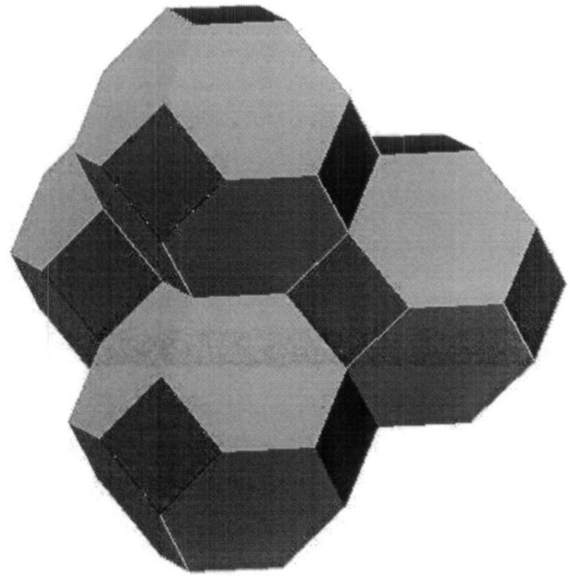

Figure 4.15

4.8.2 DIHEDRAL ANGLES OF TRUNCATED OCTAHEDRON

To find the coordinates its nodes we need to find the dihedral angles of $4_1 6_2$. It can be calculated as follows

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$4_1 6_2$

Here $n_1 = 4$ and $n_2 = 6$

We have shown that

$$\cos A_1 = \frac{2}{3} \text{ and } \cos A_2 = \frac{1}{\sqrt{6}}$$

Equation for dihedral angle is

$$\alpha_{edge} = \sum_{i,edge} \tan^{-1}\left(\frac{\cos A_i}{\sqrt{\sin^2 A_i - \cos^2 \frac{\pi}{n_i}}} \right)$$

The dihedral angle between two hexagonal(6) faces is

$$\alpha_{6-6} = 2\tan^{-1}\left(\frac{\cos A_2}{\sqrt{\sin^2 A_2 - \cos^2\frac{\pi}{6}}}\right) = 2\tan^{-1}\left(\frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{5}{6}-\frac{3}{4}}}\right)$$

$$= 2\tan^{-1}\sqrt{2} = \tan^{-1}\left(\frac{2\sqrt{2}}{1-2}\right) = \tan^{-1}(-2\sqrt{2}) = \pi - \tan^{-1} 2\sqrt{2}$$

The dihedral angle between a hexagonal(6) faces and square(4) face is

$$\alpha_{4-6} = \tan^{-1}\left(\frac{\cos A_1}{\sqrt{\sin^2 A_1 - \cos^2\frac{\pi}{4}}}\right) + \tan^{-1}\left(\frac{\cos A_2}{\sqrt{\sin^2 A_2 - \cos^2\frac{\pi}{6}}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\sqrt{\frac{5}{9}-\frac{1}{2}}}\right) + \tan^{-1}\left(\frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{5}{6}-\frac{3}{4}}}\right) = \tan^{-1} 2\sqrt{2} + \tan^{-1}\sqrt{2} = \tan^{-1}\left(\frac{2\sqrt{2}+\sqrt{2}}{1-4}\right)$$

$$= \tan^{-1}(-\sqrt{2}) = \pi - \tan^{-1}\sqrt{2}$$

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4.8.3 3D TESSELLATIONS IN FINITE ELEMENTS

We have categorized all the possible kinds of 3D space filling or tessellations using Face and vertex regular polyhedra. They were categorized as

1. Regular 3D Tessellations : 2 types(discussed).
2. Regular prism 3D Tessellations : 2 types(discussed).
3. Semi-Regular 3D Tessellations : 11 types.
4. Semi-Regular prism 3D Tessellations : 8 types.

Here we restrict overselves to regular 3D tessellations only.

In 3D finite elements it can be shown that the number of terms in 3 variable Lagrange polynomial is equal to

$$T = \sum_{r=1}^{N} {}^{3}H_r = \sum_{r=1}^{N} {}^{3+r-1}C_r = {}^{N+3}C_3 = \frac{(N+3)(N+2)(N+1)}{1.2.3}$$

The number and nature of terms are given in the following table

degree	terms correspond to	terms	partial sum	sum	cumulative sum
0	0	1	1	1	1
1	1	x, y, z	3	3	4
2	2	x^2, y^2, z^2	3	6	10
	1+1	xy, yz, zx	3		
3	3	x^3, y^3, z^3	3	10	20
	2+1	$x^2y, x^2z, y^2x, y^2z, z^2x, z^2y$	6		
	1+1+1	xyz	1		
4	4	x^4, y^4, z^4	3	15	35
	3+1	$x^3y, x^3z, y^3x, y^3z, z^3x, z^3y$	6		
	2+2	x^2y^2, x^2z^2, y^2z^2	3		
	2+1+1	x^2yz, y^2xz, z^2xy	3		
5	5	x^5, y^5, z^5	3	21	56
	4+1	$x^4y, x^4z, y^4x, y^4z, z^4x, z^4y$	6		
	3+2	$x^3y^2, x^3z^2, y^3x^2, y^3z^2, z^3x^2, z^3y^2$	6		
	3+1+1	x^3yz, y^3xz, z^3xy	3		
	2+2+1	$x^2y^2z, y^2z^2x, z^2x^2y$	3		
6	6	x^6, y^6, z^6	3	28	84
	5+1	$x^5y, x^5z, y^5x, y^5z, z^5x, z^5y$	6		
	4+2	$x^4y^2, x^4z^2, y^4x^2, y^4z^2, z^4x^2, z^4y^2$	6		
	3+3	x^3y^3, y^3z^3, z^3x^3	3		
	4+1+1	x^4yz, y^4xz, z^4xy	3		
	3+2+1	$x^3y^2z, x^3z^2y, y^3x^2z, y^3z^2x, z^3x^2y, z^3y^2x$	6		
	2+2+2	$x^2y^2z^2$	1		

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Table 4.2

4.8.4 REGULAR 3D TESSELLATIONS IN FINITE ELEMENTS

(1) Triangular Regular Prism($3_1 4_2$).

This has 6 nodes. The selected polynomial by the above criteria is $V(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 xyz + a_6 x^2 y^2 z^2$.

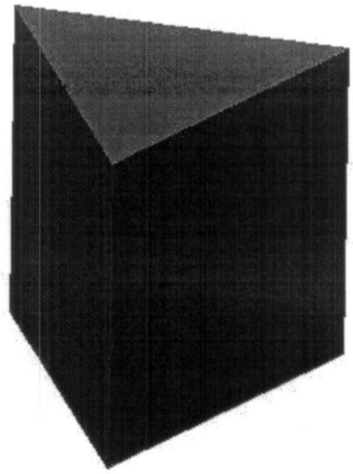

Figure 4.16

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(2) Cube(4_3).

This has 8 nodes. The selected polynomial by the above criteria is.

$$V(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 zx + a_8 xyz$$

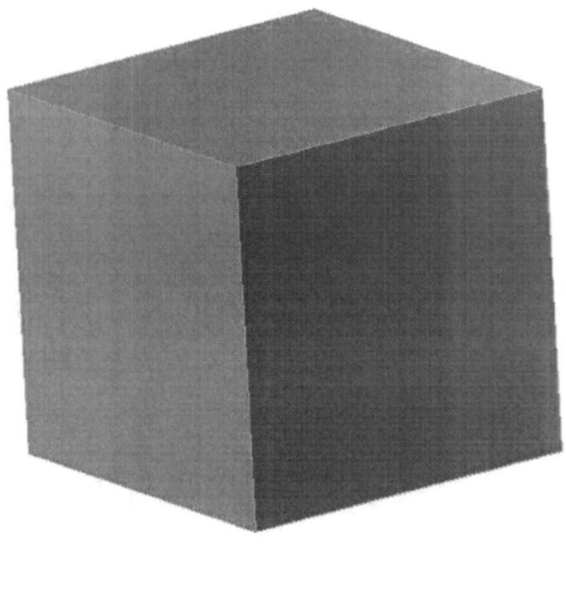

Figure 4.17

(3) Hexagonal Regular Prism($4_2 6_1$).

This has 12 nodes. The selected polynomial by the above criteria is

$$V(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx + a_{11} xyz + a_{12} x^2 y^2 z^2$$

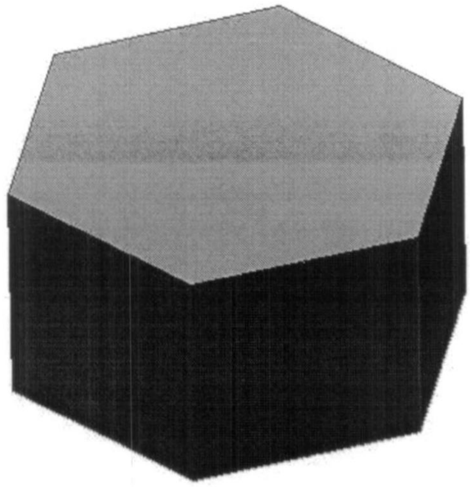

Figure 4.18

(4) Truncated Octahedron($4_1 6_2$).

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This has 24 nodes. The selected polynomial by the above criteria is

$$V(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx$$
$$+ a_{11} x^3 + a_{12} y^3 + a_{13} z^3 + a_{14} x^2 y + a_{15} x^2 z + a_{16} y^2 x + a_{17} y^2 z + a_{18} z^2 x + a_{19} z^2 y + a_{20} xyz$$
$$+ a_{21} x^2 yz + a_{22} y^2 xz + a_{23} z^2 xy + a_{24} x^2 y^2 z^2$$

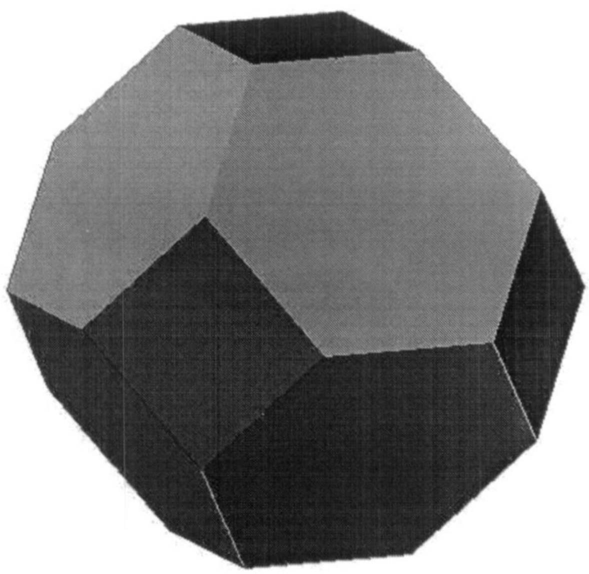

Figure 4.19

4.8.5 LIMITATIONS OF FACE AND VERTEX REGULAR POLYHEDRA AS FINITE ELEMENTS

(1) Triangular Regular Prism($3_1 4_2$).

The selected polynomial is $V(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 xyz + a_6 x^2 y^2 z^2$. For any other orientation we can transform the coordinates by $x = b_1 + b_2 X + b_3 Y + b_4 Z$, $y = c_1 + c_2 X + c_3 Y + c_4 Z$ and $z = d_1 + d_2 X + d_3 Y + d_4 Z$. But we don't obtain a similar equation.. Hence we cant predict the behavior of B using the above technique. It can be singular or non singular depending on the orientation.

Consider the following orientation with length of an edge is $2\sqrt{7}$ the coordinate set of nodes are $P = \{(0,\sqrt{7},0),(-2\sqrt{3},0,3),(0,-\sqrt{7},0),(2\sqrt{3},\sqrt{7},4),(0,0,7),(2\sqrt{3},-\sqrt{7},4)\}$.

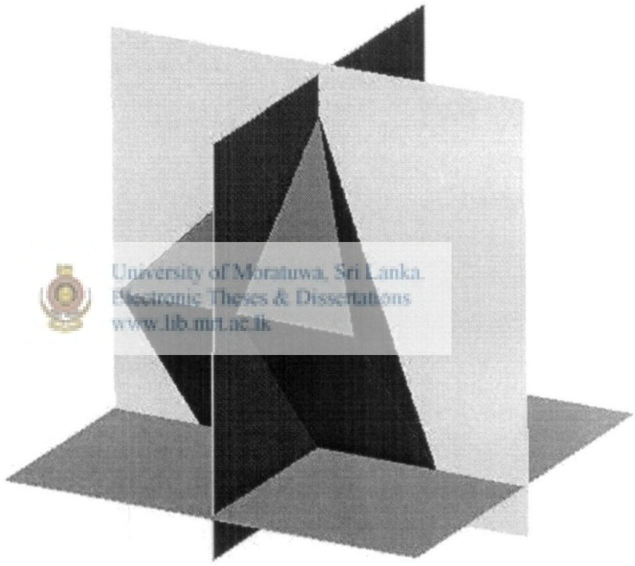

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Figure 4.20

$$\text{Here}\,|B| = \begin{vmatrix} 1 & 0 & \sqrt{7} & 0 & 0 & 0 \\ 1 & -2\sqrt{3} & 0 & 3 & 0 & 0 \\ 1 & 0 & -\sqrt{7} & 0 & 0 & 0 \\ 1 & 2\sqrt{3} & \sqrt{7} & 4 & 8\sqrt{21} & 1344 \\ 1 & 0 & 0 & 7 & 0 & 0 \\ 1 & 2\sqrt{3} & -\sqrt{7} & 4 & -8\sqrt{21} & 1344 \end{vmatrix} = -12644352 \neq 0$$

Hence Regular triangular prism can be used as a finite element in this orientation.

Consider the following orientation with length of an edge is $2\sqrt{3}$ the coordinate set of nodes are

$P = \{(0,2,\sqrt{3}),(-1,-\sqrt{3},\sqrt{3}),(1,-\sqrt{3},\sqrt{3}),(0,2,-\sqrt{3}),(-1,-\sqrt{3},-\sqrt{3}),(1,-\sqrt{3},-\sqrt{3})\}$.

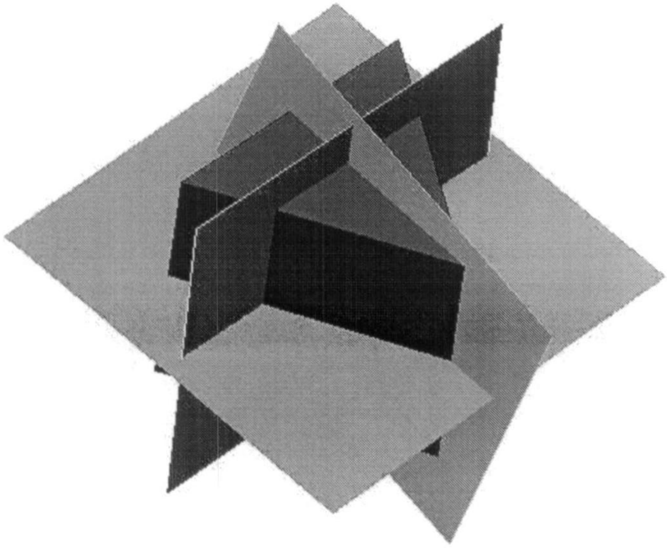

Figure 4.21

$$\text{Here}\,|B| = \begin{vmatrix} 1 & 0 & 2 & \sqrt{3} & 0 & 0 \\ 1 & -1 & -\sqrt{3} & \sqrt{3} & 3 & 9 \\ 1 & 1 & -\sqrt{3} & \sqrt{3} & -3 & 9 \\ 1 & 0 & 2 & -\sqrt{3} & 0 & 0 \\ 1 & -1 & -\sqrt{3} & -\sqrt{3} & -3 & 9 \\ 1 & 1 & -\sqrt{3} & -\sqrt{3} & 3 & 9 \end{vmatrix} = 0$$

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Hence matrix B is singular. This is because the raws are depend on each other. Triangular Regular Prism can be placed in such a way that all its nodes contained in two coordinate planes as follows.

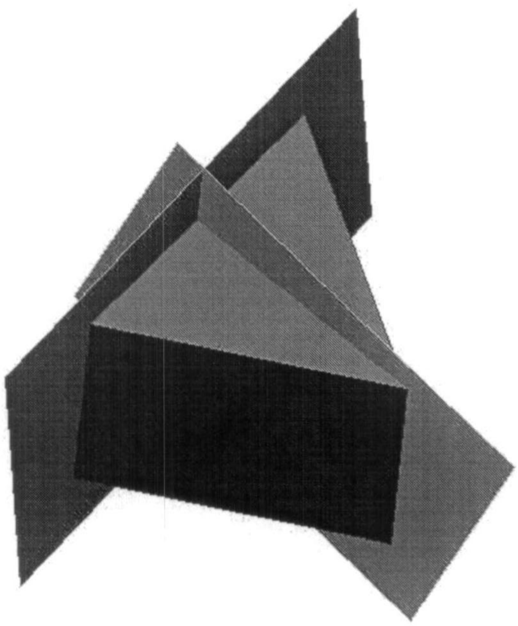

Figure 4.22

B is singular in this orientation. This is because all the nodes has at least one of x, y, z zero and the polynomial contains xyz product terms.

There are infinitely many orientations where this occurs.

An example is where coordinate set of nodes are

$P = \{(0,\sqrt{3},1),(-1,0,1),(1,0,1),(0,\sqrt{3},-1),(-1,0,-1),(1,0,-1)\}$ with length of an edge is 2

.

(2) Cube(4_3).

The selected polynomial is

$V(x,y,z) = a_1 + a_2x + a_3y + a_4z + a_5xy + a_6yz + a_7zx + a_8xyz$. For any other orientation we can transform the coordinates by $x = b_1 + b_2X + b_3Y + b_4Z$, $y = c_1 + c_2X + c_3Y + c_4Z$ and $z = d_1 + d_2X + d_3Y + d_4Z$. But we don't obtain a similar equation.. Hence we cant predict the behavior of B by the above technique. It can be singular or non singular depending on the orientation.

Consider the following orientation with length of an edge is 2 the coordinate set of nodes are $P = \{(1,1,1),(-1,1,1),(-1,-1,1),(1,-1,1),(1,1,-1),(-1,1,-1),(-1,-1,-1),(1,-1,-1)\}$.

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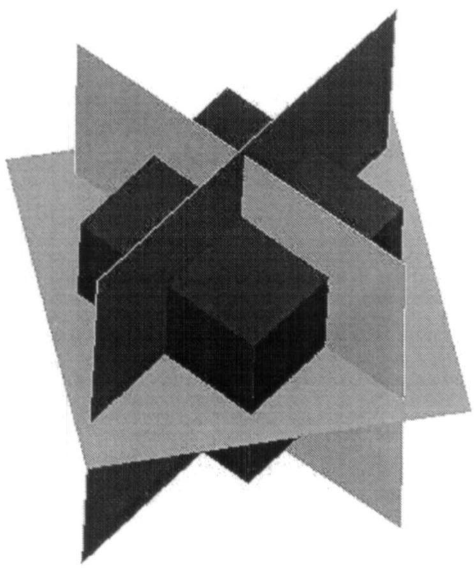

Figure 4.23

$$\text{Here } |B| = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \end{vmatrix} = 4096 \neq 0$$

Hence cube can be used as a finite element in this orientation.

Cube can be placed in such a way that all its nodes contained in two coordinate planes as follows.

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Figure 4.24

B is singular in this orientation. This is because all the nodes has at least one of x, y, z zero and the polynomial contains xyz product terms.

There are infinitely many orientations where this occurs.

An example is where coordinate set of nodes are $P = \{(1,0,1),(0,1,1),(-1,0,1),(0,-1,1),(1,0,-1),(0,1,-1),(-1,0,-1),(0,-1,-1)\}$ with length of an edge is $\sqrt{2}$.

(3) Hexagonal Regular Prism($4_2 6_1$).

The selected polynomial is

$V(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx + a_{11} xyz + a_{12} x^2 y^2 z^2$

For any other orientation we can transform the coordinates by

$x = b_1 + b_2 X + b_3 Y + b_4 Z$, $y = c_1 + c_2 X + c_3 Y + c_4 Z$ and $z = d_1 + d_2 X + d_3 Y + d_4 Z$.

But we don't obtain a similar equation. Hence we can't predict the behavior of B by the above technique. It can be singular or non singular depending on the orientation.

Consider the following situation where the coordinate set of the nodes are given by

$P = \{(4,0,3),(2,2\sqrt{3},3),(-2,2\sqrt{3},3),(-4,0,3),(-2,-2\sqrt{3},3),(2,-2\sqrt{3},3),$
$(4,0,-1),(2,2\sqrt{3},-1),(-2,2\sqrt{3},-1),(-4,0,-1),(-2,-2\sqrt{3},-1),(2,-2\sqrt{3},-1)\}$

with length of an edge is 4.

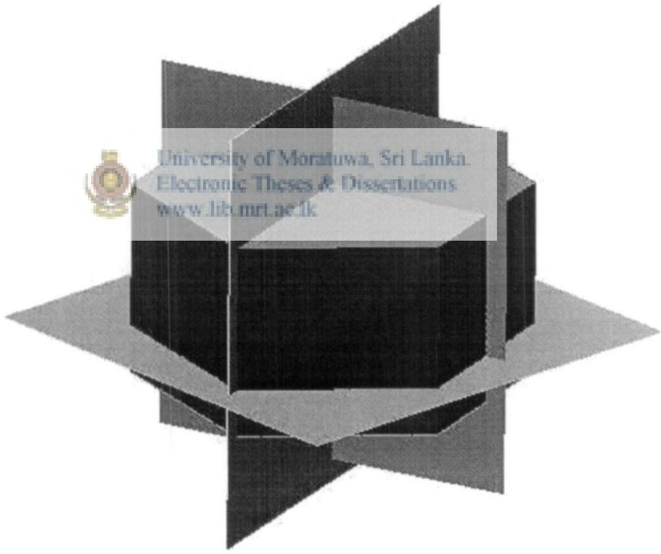
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Figure 4.25

Here

$$|B| = \begin{vmatrix} 1 & 4 & 0 & 3 & 16 & 0 & 9 & 0 & 0 & 12 & 0 & 0 \\ 1 & 2 & 2\sqrt{3} & 3 & 4 & 12 & 9 & 4\sqrt{3} & 6\sqrt{3} & 6 & 12\sqrt{3} & 432 \\ 1 & -2 & 2\sqrt{3} & 3 & 4 & 12 & 9 & -4\sqrt{3} & 6\sqrt{3} & -6 & -12\sqrt{3} & 432 \\ 1 & -4 & 0 & 3 & 16 & 0 & 9 & 0 & 0 & -12 & 0 & 0 \\ 1 & -2 & -2\sqrt{3} & 3 & 4 & 12 & 9 & 4\sqrt{3} & -6\sqrt{3} & -6 & 12\sqrt{3} & 432 \\ 1 & 2 & -2\sqrt{3} & 3 & 4 & 12 & 9 & -4\sqrt{3} & -6\sqrt{3} & 6 & -12\sqrt{3} & 432 \\ 1 & 4 & 0 & -1 & 16 & 0 & 1 & 0 & 0 & -4 & 0 & 0 \\ 1 & 2 & 2\sqrt{3} & -1 & 4 & 12 & 1 & 4\sqrt{3} & -2\sqrt{3} & -2 & -4\sqrt{3} & 48 \\ 1 & -2 & 2\sqrt{3} & -1 & 4 & 12 & 1 & -4\sqrt{3} & -2\sqrt{3} & 2 & 4\sqrt{3} & 48 \\ 1 & -4 & 0 & -1 & 16 & 0 & 1 & 0 & 0 & 4 & 0 & 0 \\ 1 & -2 & -2\sqrt{3} & -1 & 4 & 12 & 1 & 4\sqrt{3} & 2\sqrt{3} & 2 & -4\sqrt{3} & 48 \\ 1 & 2 & -2\sqrt{3} & -1 & 4 & 12 & 1 & -4\sqrt{3} & 2\sqrt{3} & -2 & 4\sqrt{3} & 48 \end{vmatrix} = 0$$

Hence Hexagonal Regular Prism cannot be used as a finite element in this orientation.

There are no orientation problems in the form of xyz terms becoming zero regarding Hexagonal Regular Prism since all its nodes cannot be contained in coordinate planes.

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Later we will show that matrix B is singular independent of the orientation which implies that Hexagonal Regular Prism can never be used as a finite element.

(4) Truncated Octahedron($4,6_2$).

The selected polynomial is

$$\begin{aligned} V(x,y,z) &= a_1 \\ &+ a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx \\ &+ a_{11} x^3 + a_{12} y^3 + a_{13} z^3 + a_{14} x^2 y + a_{15} x^2 z + a_{16} y^2 x + a_{17} y^2 z + a_{18} z^2 x + a_{19} z^2 y + a_{20} xyz \\ &+ a_{21} x^2 yz + a_{22} y^2 xz + a_{23} z^2 xy \\ &+ a_{24} x^2 y^2 z^2 \end{aligned}$$

For any other orientation we can transform the coordinates by $x = b_1 + b_2 X + b_3 Y + b_4 Z$, $y = c_1 + c_2 X + c_3 Y + c_4 Z$ and $z = d_1 + d_2 X + d_3 Y + d_4 Z$. But we don't obtain a similar equation.. Hence we cant predict the behavior of B by the above technique. It can be singular or non singular depending on the orientation.

Consider the following orientation with length of an edge is 2 the coordinate set of nodes are

$P = \{(3,-1,0),(3,1,0),(1,3,0),(-1,3,0),(-3,1,0),(-3,-1,0),(-1,-3,0),(1,-3,0),$
$(1,1,2\sqrt{2}),(1,1,-2\sqrt{2}),(-1,1,-2\sqrt{2}),(-1,1,2\sqrt{2}),(1,-1,2\sqrt{2}),(1,-1,-2\sqrt{2}),$
$(-1,-1,-2\sqrt{2}),(-1,-1,2\sqrt{2}),$
$(2,-2,\sqrt{2}),(2,2,\sqrt{2}),(-2,2,\sqrt{2}),(-2,-2,\sqrt{2}),(2,-2,-\sqrt{2}),(2,2,-\sqrt{2}),(-2,2,-\sqrt{2}),(-2,-2,-\sqrt{2})\}$

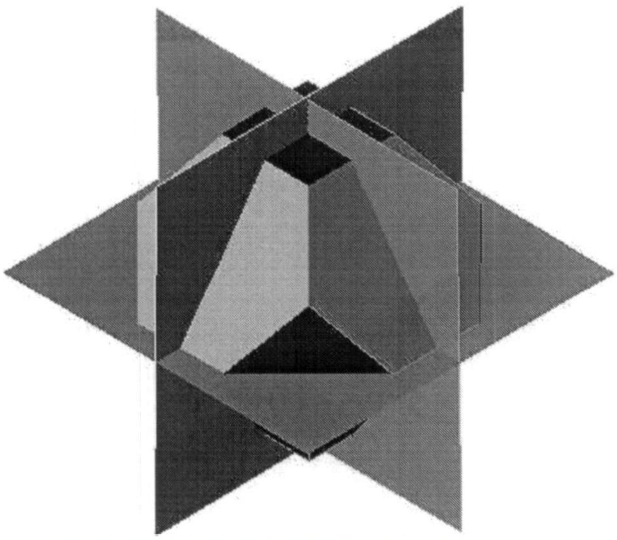

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Figure4.26

The matrix B is as follows

$$
|B| = \begin{vmatrix}
1 & 3 & -1 & 0 & 9 & 1 & 0 & -3 & 0 & 0 & 27 & -1 & 0 & -9 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 9 & 1 & 0 & 3 & 0 & 0 & 27 & 1 & 0 & 9 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 & 1 & 9 & 0 & 3 & 0 & 0 & 1 & 27 & 0 & 3 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 3 & 0 & 1 & 9 & 0 & -3 & 0 & 0 & -1 & 27 & 0 & 3 & 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 9 & 1 & 0 & -3 & 0 & 0 & -27 & 1 & 0 & 9 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & -1 & 0 & 9 & 1 & 0 & 3 & 0 & 0 & -27 & -1 & 0 & -9 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -3 & 0 & 1 & 9 & 0 & 3 & 0 & 0 & -1 & -27 & 0 & -3 & 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -3 & 0 & 1 & 9 & 0 & -3 & 0 & 0 & 1 & -27 & 0 & -3 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2\sqrt{2} & 1 & 1 & 8 & 1 & 2\sqrt{2} & 2\sqrt{2} & 1 & 1 & 16\sqrt{2} & 1 & 2\sqrt{2} & 1 & 2\sqrt{2} & 8 & 8 & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 8 & 8 \\
1 & 1 & 1 & -2\sqrt{2} & 1 & 1 & 8 & 1 & -2\sqrt{2} & -2\sqrt{2} & 1 & 1 & -16\sqrt{2} & 1 & -2\sqrt{2} & 1 & -2\sqrt{2} & 8 & 8 & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & 8 & 8 \\
1 & -1 & 1 & -2\sqrt{2} & 1 & 1 & 8 & -1 & -2\sqrt{2} & 2\sqrt{2} & -1 & 1 & -16\sqrt{2} & 1 & -2\sqrt{2} & -1 & -2\sqrt{2} & -8 & 8 & 2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} & -8 & 8 \\
1 & -1 & 1 & 2\sqrt{2} & 1 & 1 & 8 & -1 & 2\sqrt{2} & -2\sqrt{2} & -1 & 1 & 16\sqrt{2} & 1 & 2\sqrt{2} & -1 & 2\sqrt{2} & -8 & 8 & -2\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} & -8 & 8 \\
1 & 1 & -1 & 2\sqrt{2} & 1 & 1 & 8 & -1 & -2\sqrt{2} & 2\sqrt{2} & 1 & -1 & 16\sqrt{2} & -1 & 2\sqrt{2} & 1 & 2\sqrt{2} & 8 & -8 & -2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} & -8 & 8 \\
1 & 1 & -1 & -2\sqrt{2} & 1 & 1 & 8 & -1 & 2\sqrt{2} & -2\sqrt{2} & 1 & -1 & -16\sqrt{2} & -1 & -2\sqrt{2} & 1 & -2\sqrt{2} & 8 & -8 & 2\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} & -8 & 8 \\
1 & -1 & -1 & -2\sqrt{2} & 1 & 1 & 8 & 1 & 2\sqrt{2} & 2\sqrt{2} & -1 & -1 & -16\sqrt{2} & -1 & -2\sqrt{2} & -1 & -2\sqrt{2} & -8 & -8 & -2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 8 & 8 \\
1 & -1 & -1 & 2\sqrt{2} & 1 & 1 & 8 & 1 & -2\sqrt{2} & -2\sqrt{2} & -1 & -1 & 16\sqrt{2} & -1 & 2\sqrt{2} & -1 & 2\sqrt{2} & -8 & -8 & 2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & 8 & 8 \\
1 & 2 & -2 & \sqrt{2} & 4 & 4 & 2 & -4 & -2\sqrt{2} & 2\sqrt{2} & 8 & -8 & 2\sqrt{2} & -8 & 4\sqrt{2} & 8 & 4\sqrt{2} & 4 & -4 & -4\sqrt{2} & -8\sqrt{2} & 8\sqrt{2} & -8 & 32 \\
1 & 2 & 2 & \sqrt{2} & 4 & 4 & 2 & 4 & 2\sqrt{2} & 2\sqrt{2} & 8 & 8 & 2\sqrt{2} & 8 & 4\sqrt{2} & 8 & 4\sqrt{2} & 4 & 4 & 4\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} & 8 & 32 \\
1 & -2 & 2 & \sqrt{2} & 4 & 4 & 2 & -4 & 2\sqrt{2} & -2\sqrt{2} & -8 & 8 & 2\sqrt{2} & 8 & 4\sqrt{2} & -8 & 4\sqrt{2} & -4 & 4 & -4\sqrt{2} & 8\sqrt{2} & -8\sqrt{2} & -8 & 32 \\
1 & -2 & -2 & \sqrt{2} & 4 & 4 & 2 & 4 & -2\sqrt{2} & -2\sqrt{2} & -8 & -8 & 2\sqrt{2} & -8 & 4\sqrt{2} & -8 & 4\sqrt{2} & -4 & -4 & 4\sqrt{2} & -8\sqrt{2} & -8\sqrt{2} & 8 & 32 \\
1 & 2 & -2 & -\sqrt{2} & 4 & 4 & 2 & -4 & 2\sqrt{2} & -2\sqrt{2} & 8 & -8 & -2\sqrt{2} & -8 & -4\sqrt{2} & 8 & -4\sqrt{2} & 4 & -4 & 4\sqrt{2} & 8\sqrt{2} & -8\sqrt{2} & -8 & 32 \\
1 & 2 & 2 & -\sqrt{2} & 4 & 4 & 2 & 4 & -2\sqrt{2} & -2\sqrt{2} & 8 & 8 & -2\sqrt{2} & 8 & -4\sqrt{2} & 8 & -4\sqrt{2} & 4 & 4 & -4\sqrt{2} & -8\sqrt{2} & -8\sqrt{2} & 8 & 32 \\
1 & -2 & 2 & -\sqrt{2} & 4 & 4 & 2 & -4 & -2\sqrt{2} & 2\sqrt{2} & -8 & 8 & -2\sqrt{2} & 8 & -4\sqrt{2} & -8 & -4\sqrt{2} & -4 & 4 & 4\sqrt{2} & -8\sqrt{2} & 8\sqrt{2} & -8 & 32 \\
1 & -2 & -2 & -\sqrt{2} & 4 & 4 & 2 & 4 & 2\sqrt{2} & 2\sqrt{2} & -8 & -8 & -2\sqrt{2} & -8 & -4\sqrt{2} & -8 & -4\sqrt{2} & -4 & -4 & -4\sqrt{2} & 8\sqrt{2} & 8\sqrt{2} & 8 & 32
\end{vmatrix} = 0
$$

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Truncated Octahedron can be placed in such a way that all its nodes contained the three coordinate planes as follows.

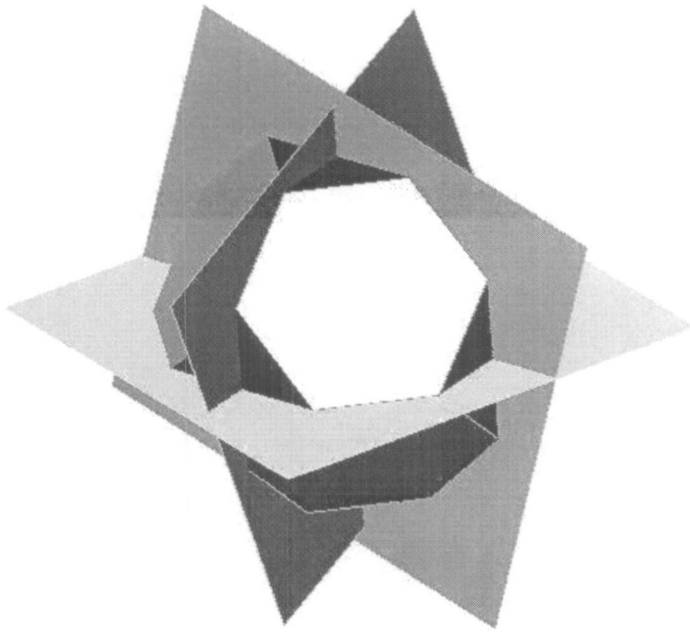

Figure 4.27

B is singular in this orientation. This is because all the nodes has at least one of x, y, z zero and the polynomial contains xyz product terms.

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But there is only one orientation where this occurs with coordinate set of nodes are

$$P = \{(1,0,2),(0,1,2),(-1,0,2),(0,-1,2),(1,0,-2),(0,1,-2),(-1,0,-2),(0,-1,-2),$$
$$(1,2,0),(0,2,1),(-1,2,0),(0,2,-1),(1,-2,0),(0,-2,1),(-1,-2,0),(0,-2,-1),$$
$$(2,1,0),(2,0,1),(2,-1,0),(2,0,-1),(-2,1,0),(-2,0,1),(-2,-1,0),(-2,0,-1)\}$$

if length of an edge is $\sqrt{2}$.

Later we will show that matrix B is singular independent of the orientation which implies that Hexagonal Regular Prism can never be used as a finite element.

4.8.6 PROOF OF A GENERAL RESULT

1.Assume that the selected 3D Lagrange polynomial for the polyhedron includes the 2D complete polynomial of degree 2.

2.Suppose we have a polygonal face with 6 or more number of sides in the polyhedron.

3.We select the nodes(points where function values are assumed) at vertices.

4.We select the X,Y,Z coordinate system in such a way that the polygon is confined to XY plane. Thus $Z = 0$ for all the vertices.

5.We can transform the nodes to x,y,z coordinate system by $x = b_1 + b_2X + b_3Y + b_4Z$, $y = c_1 + c_2X + c_3Y + c_4Z$ and $z = d_1 + d_2X + d_3Y + d_4Z$.

6. Since $Z = 0$, this reduces to $x = b_1 + b_2X + b_3Y$, $y = c_1 + c_2X + c_3Y$ for x,y,z coordinates.

7. Thus the situation is similar to that discussed under 2D finite element.

8. Hence $|f(P')| = 0$ where $P' = (x,y,z)$ is the coordinate set of vertices.

9. Hence we can't use the above polyhedron as a finite element.

4.8.7 CONCLUSION

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Any polyhedron having a polygonal face with two axis of symmetry and having six or more number of vertices with the nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains a two variable complete polynomial of degree two.

4.8.8 DEDUCTIONS

(1) Among face and vertex regular polyhedra with nodes ≥ 10 (will automatically contain the two variable complete polynomial of degree 2) and having a face with ≥ 6 vertices cannot be used as finite element. So the only possible polyhera that can be used as finite elements in 3D are as follows. Some of them cannot fill space.

1. 4_3 – *Cube*
2. $3_1 4_2$ – *Triangular Regular Prism*
3. 3_3 – *Tetrahedron*
4. 5_3 – *Dodecahedron*

5. 3_4 – *Octahedron*

6. 3_5 – *Icosahedron*

7. $3_1 4_3$ – *Small Rhombicuboctahedron*

8. $3_2 4_2$ – *Cuboctahedron*

9. $3_2 5_2$ – *Icosidodecahedron*

10. $3_1 4_2 5_1$ – *Small Rhombicosidodecahedron*

11. $3_4 4_1$ – *Snub Cube*

12. $3_4 5_1$ – *Snub Dodecahedron*

13. $4_2 5_1$ – *Pentagonal Regular Prism*

14. $3_3 5_1$ – *Pentagonal Regular Anti Prism*

(2) Hexagonal Regular Prism and Truncated Octahedron cannot be used as finite elements in 3D with the selected polynomial since they contain regular hexagonal faces and the 3D Lagrange polynomial contains the corresponding 2D complete polynomial of degree 2.

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Therefore the only possible regular 3D tessellations for finite elements are $(3_1 4_2)_{12}$ and $(4_3)_8$.

The corresponding finite elements are **Triangular Regular Prism** and **Cube.**

(3) Other 3D tessellations which can be used in finite element analysis are

1. $(3_3)_8(3_4)_6$
2. $(3_4)_2(3_2 4_2)_4$
3. $(4_3)_2(3_2 4_2)_1(3_1 4_3)_2$
4. $(3_3)_1(4_3)_1(3_1 4_3)_3$
5. $(3_1 4_2)_6(4_3)_4$
6. $(3_1 4_2)_4(4_3)_2(3_1 4_2)_2(4_3)_2$

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

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CONCLUSIONS

(1) Following criteria is proposed for defining the piecewise Lagrange polynomial
1. Select the complete polynomial of immediate lesser number of terms.
2. Select the other terms from the immediate symmetric higher degree terms.
3. When there is more than one possibility always select terms with more types of product terms.

(2) With the piecewise polynomial selected in the above manner the only possible regular tessellations for finite elements are
2D-Equilateral Triangle, Square
3D-Regular Triangular Prism, Cube

(3) Any polygon having two axis of symmetry with nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains the complete polynomial of degree two

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(4) Any polyhedron having a polygonal face with two axis of symmetry and having six or more number of vertices with the nodes are selected at vertices cannot be used as a finite element if its Lagrange polynomial contains a two variable complete polynomial of degree two

(5) Radius(R) of the escribed sphere of a face and vertex regular polyhedron in which M_i number of polygons of n_i number of sides of length a meet, satisfies

$$R = \frac{a}{2}\frac{1}{\sqrt{1-\left(\dfrac{\cos\dfrac{\pi}{n_i}}{\sin A_i}\right)^2}} \quad \text{where } \sum_i M_i A_i = \pi$$

(6) Sphere is a limiting case of a polyhedron.

RECOMMENDATIONS

It is recommended to carryout an investigation to find out the fourth and higher dimensional regular polytopes and tessellations.

Analyzing these combinations for finite elements will be useful in solving the partial differential equations of four or higher variables.

It is also recommended to study the criteria of selecting the second and higher order piecewise polynomial which will define a way to avoid the possible non existing ones.

The result that the sphere can be treated as a polyhedron can be used for finite element analysis on surfaces.

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APPENDIX

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APPENDIX A

PROOF OF THE SINE FORMULA IN SPHERICAL TRIGONOMETRY

Let ABC be a spherical triangle on the surface of a sphere with centre G.

Let the perpendicular drop from the vertex D meet the plane ABG at point C. Lets complete the right triangle triangles CDE and CFD as in the figure A.1.

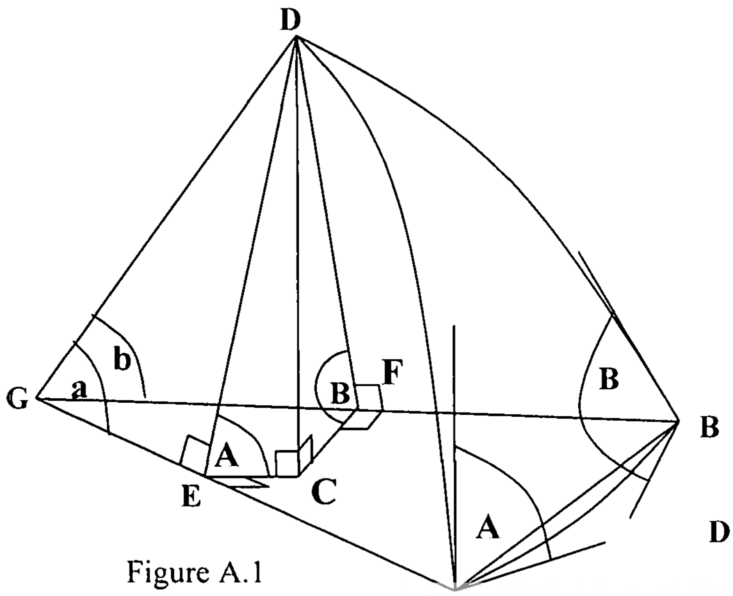

Figure A.1

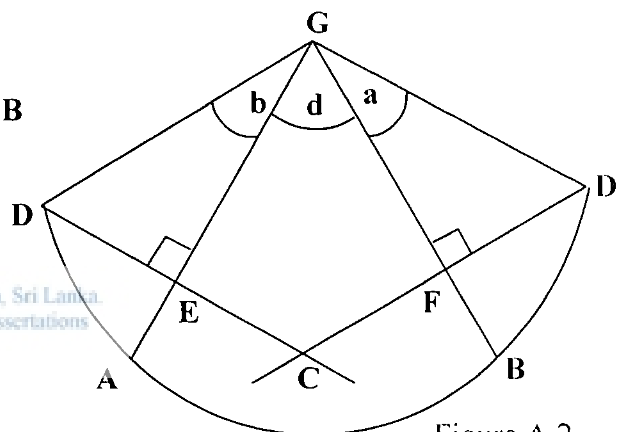

Figure A.2

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From figure A.1 we have

$$\frac{CD}{DF} = \sin B \text{ - - - - - - - }(1) \quad \frac{CD}{DE} = \sin A \text{ - - - - - - - }(2)$$

When the spherical triangle is flattened on a surface we get the figureA.2. So we have

$$\frac{DE}{DG} = \sin b \text{ - - - }(3) \quad \frac{DF}{DG} = \sin a \text{ - - - - - - }(4)$$

From (1), (2), (3), (4)

$$\frac{DE.DF}{DG.CD} = \frac{\sin b}{\sin B} = \frac{\sin a}{\sin A} \text{ - - - - - - }(5)$$

Similarly by drawing perpendiculars from vertex B to the plane ADG we have

$$\frac{\sin d}{\sin D} = \frac{\sin a}{\sin A} \text{ - - - - - - - - }(6)$$

$(5),(6) \Rightarrow$

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin d}{\sin D}$$

APPENDIX B

SOLUTION OF THE CUBIC EQUATION

Let the general cubic equation be $ax^3 + bx^2 + cx + d = 0 \quad ; a \neq 0$.

By letting $x = y + r$ and choosing $3ar + b = 0$ the original equation can be transferred to $y^3 - 3py - q = 0 --------(1)$.

Now since

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$(u+v)^3 - 3uv(u+v) - (u^3 + v^3) = 0$$

We choose $y = u + v$ hence

$$y^3 - 3uvy - (u^3 + v^3) = 0 ------- (2)$$

We compare the coefficients of (1) and (2)

$$-3uv = -3p \Rightarrow p = uv --------(3)$$

$$-q = -(u^3 + v^3) \Rightarrow q = u^3 + v^3 ----(4)$$

By eliminating v from (3) and (4)

$$q = u^3 + \left(\frac{p}{u}\right)^3$$

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$$\Rightarrow (u^3)^2 - q(u^3) + p^3 = 0$$

$$\Rightarrow u^3 = \frac{q \pm \sqrt{q^2 - 4p^3}}{2}$$

$$(4) \Rightarrow v^3 = q - u^3 = \frac{q \mp \sqrt{q^2 - 4p^3}}{2}$$

We choose

$$u^3 = \frac{q + \sqrt{q^2 - 4p^3}}{2} \Rightarrow u = \left(\frac{q + \sqrt{q^2 - 4p^3}}{2}\right)^{\frac{1}{3}}$$

$$v^3 = \frac{q - \sqrt{q^2 - 4p^3}}{2} \Rightarrow v = \left(\frac{q - \sqrt{q^2 - 4p^3}}{2}\right)^{\frac{1}{3}}$$

So the final solution is

$$x = y + r = u + v + r = \left(\frac{q + \sqrt{q^2 - 4p^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{q - \sqrt{q^2 - 4p^3}}{2}\right)^{\frac{1}{3}} + r$$

APPENDIX C
SOLID ANGLE OF A VERTEX

This is the internal solid angle of a vertex of the polyhedron. Due to the similarity of its vertices this is a constant for face and vertex regular polyhedra.
Consider the following spherical triangle BEF.

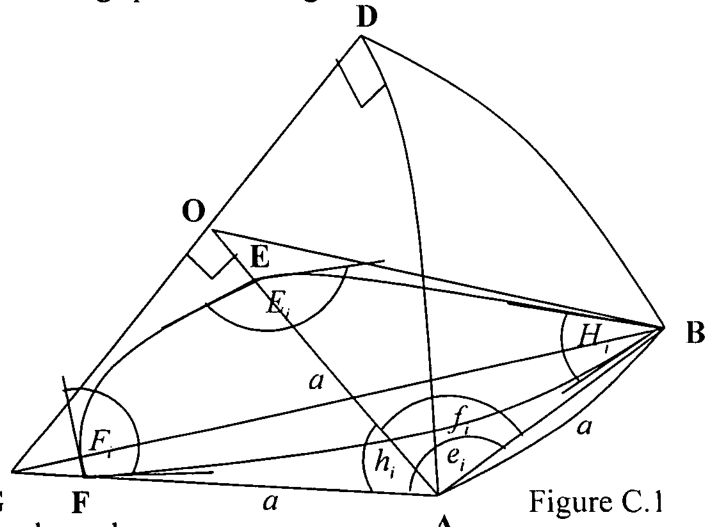

Figure C.1

$$E_i = \frac{\pi}{2}$$
$$F_i = A_i$$
$$e_i = \frac{\pi}{2} - \frac{d_i}{2}$$
$$f_i = \frac{\pi}{2} - \frac{\pi}{n_i}$$
$$h_i = \frac{\pi}{2} - b_i$$

By the sin formula we have

$$\frac{\sin E_i}{\sin e_i} = \frac{\sin F_i}{\sin f_i} = \frac{\sin H_i}{\sin h_i} \Rightarrow \frac{1}{\cos\frac{d_i}{2}} = \frac{\sin A_i}{\cos\frac{\pi}{n_i}} = \frac{\sin H_i}{\cos b_i} = \frac{1}{k}$$

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$$\therefore \sin^2 H_i = \frac{\cos^2 b_i}{k^2} = \frac{\cos^2 a_i}{k^2} = \frac{1-\sin^2 a_i}{k^2} = \frac{1-\frac{a^2}{4R^2}\operatorname{cosec}^2\frac{\pi}{n_i}}{k^2} = \frac{1-(1-k^2)\operatorname{cosec}^2\frac{\pi}{n_i}}{k^2}$$

$$= \operatorname{cosec}^2\frac{\pi}{n_i} - \frac{\cot^2\frac{\pi}{n_i}}{k^2} = \frac{1-\frac{\cos^2\frac{\pi}{n_i}}{k^2}}{\sin^2\frac{\pi}{n_i}} = \frac{1-\sin^2 A_i}{\sin^2\frac{\pi}{n_i}} = \left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right)^2 \Rightarrow \sin H_i = \frac{\cos A_i}{\sin\frac{\pi}{n_i}}$$

Area of the spherical triangle BEF is

$$A = a^2(E_i + F_i + H_i - \pi) = a^2\left(\frac{\pi}{2} + A_i + H_i - \pi\right) = a^2\left(A_i + H_i - \frac{\pi}{2}\right)$$

The solid angle due to the ith type of polygon is

$$\omega_i = 2M_i\frac{A}{a^2} = 2M_i\left(A_i + H_i - \frac{\pi}{2}\right)_i = 2M_iA - \pi M_i + 2M_iH_i$$

Total solid angle is

$$\omega = \sum_i \omega_i = 2\sum_i M_iA_i - \pi\sum_i M_i + 2\sum_i M_iH_i = 2\pi - \pi\sum_i M_i + 2\sum_i M_iH_i$$

$$= 2\pi - \pi\sum_i M_i + 2\sum_i M_i \sin^{-1}\left(\frac{\cos A_i}{\sin\frac{\pi}{n_i}}\right)$$

APPENDIX D
DIHEDRAL ANGLE

Dihedral angle is the plane angle between faces of the polyhedron. This can be calculated by considering the spherical triangle as in the earlier calculations.

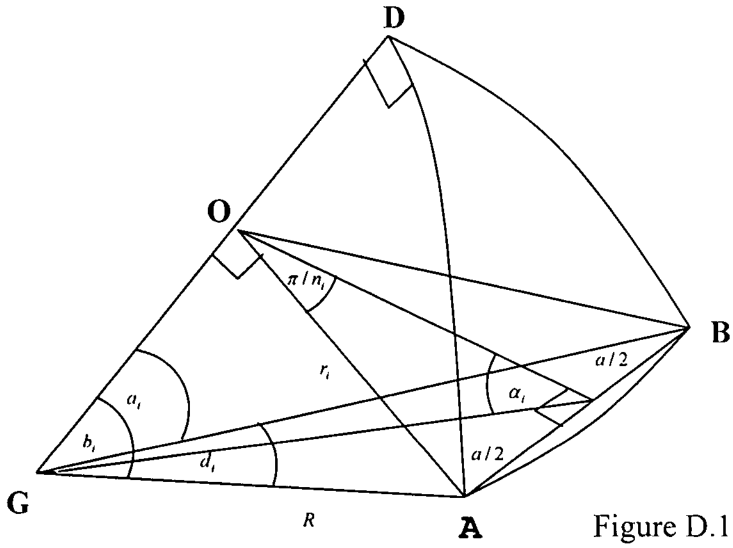

Figure D.1

$$\sin a_i = \frac{a}{2R\sin\frac{\pi}{n_i}}$$

$$\tan\alpha_i = \frac{r_i\cot a_i}{r_i\cos\frac{\pi}{n_i}}$$

$$\tan^2\alpha_i$$

$$=\cot^2 a_i \sec^2\frac{\pi}{n_i}$$

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$$=\left(\frac{4R^2\sin^2\frac{\pi}{n_i}}{a^2}-1\right)\sec^2\frac{\pi}{n_i}=\left(\frac{4\sin^2\frac{\pi}{n_i}}{a^2}\frac{a^2}{4}\frac{1}{1-\frac{\cos^2\frac{\pi}{n_i}}{\sin^2 A_i}}-1\right)\sec^2\frac{\pi}{n_i}$$

$$=\frac{\sin^2\frac{\pi}{n_i}\sin^2 A_i-\sin^2 A_i+\cos^2\frac{\pi}{n_i}}{\sin^2 A_i-\cos^2\frac{\pi}{n_i}}\sec^2\frac{\pi}{n_i}=\frac{\cos^2\frac{\pi}{n_i}\left(1-\sin^2 A_i\right)}{\sin^2 A_i-\cos^2\frac{\pi}{n_i}}\sec^2\frac{\pi}{n_i}$$

$$=\frac{\cos^2 A_i}{\sin^2 A_i-\cos^2\frac{\pi}{n_i}}$$

Therefore the dihedral angle is

$$\alpha_{edge}=\sum_{i,edge}\alpha_i=\sum_{i,edge}\tan^{-1}\left(\frac{\cos A_i}{\sqrt{\sin^2 A_i-\cos^2\frac{\pi}{n_i}}}\right)$$

APPENDIX E

FACES, VERTICES AND EDGES

n_i = number of edges(= vertices) of the i th type polygon

M_i = number of i th type polygons meet at a vertex of the polyhedron

N_i = number of i th type polygons in the polyhedron

F = total number of faces in the polyhedron

V = total number of vertices in the polyhedron

E = total number of edges in the polyhedron

As every vertex is identical and each polygon type contributes to the vertices the number of vertices can be calculated by considering only one type of a polygon. i.e

$$V = \frac{n_i N_i}{M_i}$$

Two edges of polygons produce one edge of the polyhedron

$$E = \frac{\sum_i n_i N_i}{2} = \frac{V \sum_i M_i}{2}$$

Total number of faces is

$$F = \sum_i N_i = V \sum_i \frac{M_i}{n_i}$$

By substituting these in the Euler's equation [11]

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$$F + V = 2 + E$$

$$V \sum_i \frac{M_i}{n_i} + V = 2 + \frac{V \sum_i M_i}{2}$$

$$\Rightarrow V = \frac{2}{1 + \sum M_i \left(\frac{1}{n_i} - \frac{1}{2} \right)}$$

The total number of edges is

$$E = \frac{V \sum_i M_i}{2} = \frac{\sum_i M_i}{1 + \sum M_i \left(\frac{1}{n_i} - \frac{1}{2} \right)}$$

The total number of faces is

$$F = V \sum_i \frac{M_i}{n_i} = \frac{2 \sum_i \frac{M_i}{n_i}}{1 + \sum M_i \left(\frac{1}{n_i} - \frac{1}{2} \right)}$$

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