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Conditioning and updating evidence

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Abstract

A new interpretation of Dempster–Shafer conditional notions based *directly* upon the mass assignments is provided. The masses of those propositions that *may* imply the complement of the conditioning proposition are shown to be completely annulled by the conditioning operation; conditioning may then be construed as a re-distribution of the masses of some of these propositions to those that *definitely* imply the conditioning proposition. A complete characterization of the propositions whose masses are annulled without re-distribution, annulled with re-distribution and enhanced by the re-distribution of masses is provided. A new evidence updating strategy that is composed of a linear combination of the available evidence and the conditional evidence is also proposed. It enables one to account for the 'integrity' and 'inertia' of the available evidence and its 'flexibility' to updating by appropriate selection of the linear combination weights. Several such strategies, including one that has a probabilistic interpretation, are also provided.

Keywords: Evidential reasoning; Dempster–Shafer theory; Distributed decision networks; Conditioning evidence; Updating evidence

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1. Introduction

Methods of representing and dealing with uncertainty in artificial intelligence have received considerable attention for several decades. Among the many symbolic and numerical methods that have been proposed, the theory of belief functions, also known as Dempster–Shafer (DS) theory, has gained increasing recognition as a framework capable of representing and manipulating uncertain and partial knowledge. The work of Dempster [1] and Shafer [2] has led to a large number of important theoretical contributions in belief function theory in the last two decades [3–10]. This theory has been successfully applied in target tracking and identification [11], robotics [12,13], map building [14,15], document retrieval [16], computer vision [17], pattern classification [18], automated task recognition [19], data mining [20–22], business and economics [23], to name a few. The main advantage of DS theory lies in its ability to numerically quantify the lack of knowledge in an effective manner.

Updating or conditioning a body of evidence [24,25] modeled within the DS framework prior to the availability of a particular piece of information plays an important role in most of these applications. For example, consider a distributed decision network that has been deployed in a battlefield. These types of distributed decision-making environments have generated tremendous interest in recent years due to their wide scope of applicability [26]. A typical decision node in the hierarchy of such a network performs fusion of the information or data it obtains from its child nodes in the lower levels of the hierarchy. Suppose this information from its child sensor nodes has enabled a decision node to form a knowledge base regarding the location of certain objects in the battlefield. A DS modeling framework allows this to be carried out via a suitable basic belief assignment on, for example, an appropriately defined grid [15]. Suppose then the node receives a new piece of evidence, from perhaps a mobile robot or ground troops, that a particular enemy object has been destroyed; hence the grid position previously occupied by this object is now vacant. Clearly, the decision node now has to update its assignments in light of this new evidence.

How 'willing' or 'flexible' the decision node is for updating depends on its perceived 'reliability' of the source providing the new information. In situations where the original knowledge base had been constructed from a vast amount of evidence gathered from past experience and/or numerous experts, it may be reluctant to compromise the 'integrity' of its knowledge base and its 'inertia' should not be ignored when updating is warranted. Various strategies for updating evidence have been proposed over the years [24,25]. However, these strategies do not appear to provide a convenient method to account for the integrity and inertia of the evidence and its flexibility to updating.

In this paper, we first provide a new interpretation of DS conditional notions based *directly* upon the corresponding mass assignments thus providing a more intuitive interpretation of the conditioning operation and how it impacts the remaining propositions. A new evidence updating strategy conditional to a given proposition is then proposed. The strategy we propose is a linear combination of the available evidence and the conditional evidence. The integrity and inertia of the available evidence, and its flexibility to updating, can then be accounted for via these linear combination weights. Various strategies to choose these linear combination weights are also proposed. One such strategy, for which a probabilistic interpretation is provided, we believe is quite novel and is expected to be extremely useful when no specific information regarding the reliability of the incoming new evidence is available.

2. Preliminaries

Reals are denoted by $\Re$. Given the interval $[a,b]$ in $\Re$, $\ell[a,b]$ denotes its length, viz., $\ell[a,b] = b - a$; its scalar multiplication is $K \cdot [a,b] = [Ka, Kb]$. The addition of intervals $[a_1, b_1]$ and $[a_2, b_2]$ in $\Re$ are defined as $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$. For set Θ, $|\Theta|$ denotes its cardinality; for $a \in \Re$, $|a|$ denotes its absolute value.

2.1. Introduction to DS theory

DS theory [1,2] was motivated by various concerns including dissatisfaction with certain axioms in probability. In the standard probability framework all objects in the sample space are assigned a probability. Any object for which there is no information is assigned an equal a-priori probability. Hence when the degree of support for an event is known, the remainder of the support is automatically assigned to the negation of the event. On the other hand, in DS theory, mass assignments are carried out for the events, or propositions as they are known, in the sample space. The mass assignments to non-singleton propositions generate a notion of uncertainty. Objects for which there is no information are not assigned an a-priori mass. Hence committing support for an event does not necessarily imply that the remaining support is committed to its negation; the lack of support for any particular event simply implies support for all other events. In other words, the additivity axiom in the probability formalism is relaxed in DS theory.

We denote the total set of mutually exclusive and exhaustive objects or singletons via $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$. In DS theory, Θ is referred to as the *frame of discernment* (FoD), or simply the *frame*, signifying the scope of our objective. A singleton represents the lowest level of information that is discernible by the system. Given $|\Theta| = n$, the power set of Θ denoted by 2^Θ, contains 2^n elements that are composed of all the subsets of Θ. The elements in 2^Θ form the propositions of interest in DS theory; therefore the mass assigned to a proposition is free to move into the individual singleton objects that form the composite proposition thus generating the notion of *ignorance*. The support for any such proposition is provided via a *basic belief assignment* (BBA).

Definition 1 (*Basic belief assignment* (*BBA*)). The mapping $m : 2^{\Theta} \mapsto [0, 1]$ is a BBA for the frame Θ if: (i) $m(\emptyset) = 0$; and (ii) $\sum_{A\subseteq\Theta} m(A) = 1$.

The set of propositions in a frame Θ that possess nonzero BBAs or masses are called the *focal elements* of Θ; it is denoted by $\mathscr{F}(\Theta) = \{A \subseteq \Theta : m(A) > 0\}$. The triple $\{\Theta, \mathscr{F}, m\}$ is referred to as the corresponding *body of evidence* (BoE).

The quantity $m(A)$ measures the support assigned to proposition A *only*; the belief assigned to A on the other hand must take into account the supports for all proper subsets of A as well.

Definition 2 (*Belief*). Given a BoE $\{\Theta, \mathscr{F}, m\}$, the *belief* assigned to $A \subseteq \Theta$ is Bel: $2^{\Theta} \mapsto [0, 1]$ where $\mathrm{Bel}(A) = \sum_{B\subseteq A} m(B)$.

We use the notation $\widehat{\mathscr{F}}(\Theta)$ to denote those propositions in a frame Θ that possess nonzero beliefs, viz., i$\widehat{\mathscr{F}}(\Theta) = \{A \subseteq \Theta : \mathrm{Bel}(A) > 0\}$. As will be evident later, propositions are conditioned with respect to propositions in $\widehat{\mathscr{F}}(\Theta)$ only.

$\mathrm{Bel}(A)$ represents the total support that can move into A without any ambiguity. It can be characterized without reference to the underlying BBA via.

Theorem 3 [2]. *For a given FoD* Θ, *the function* $\mathrm{Bel} : 2^{\Theta} \mapsto [0, 1]$ *constitutes a belief function iff* (*i*) $\mathrm{Bel}(\emptyset) = 0$; (*ii*) $\mathrm{Bel}(\Theta) = 1$; *and* (*iii*) *for every collection* $\{A_i\}_{i=\overline{1,n}}$, $A_i \subseteq \Theta$

$$\mathrm{Bel}\left(\bigcup_{i=\overline{1,n}} A_i\right) \geqslant \sum_{\substack{I\subseteq\{1,\ldots,n\}\\ I\neq\emptyset}} (-1)^{|I|+1}\mathrm{Bel}\left(\bigcap_{i\in I} A_i\right). \tag{1}$$

The *relative complement* of proposition A with respect to $X \subseteq \Theta$, denoted by $X - A$, consists of all singletons not included in, and not implying, A, viz., $X - A = \{\theta : \theta \in X, \theta \notin A\}$. We use $\bar{A}$ to denote $\Theta - A$.

Now we may quantify the extent to which one doubts a proposition.

Definition 4 (*Doubt*). Given a BoE $\{\Theta, \mathscr{F}, m\}$, the *doubt* regarding $A \subseteq \Theta$ is $\mathrm{Dou} : 2^{\Theta} \mapsto [0, 1]$ where $\mathrm{Dou}(A) = \mathrm{Bel}(\bar{A})$.

With Definitions 2 and 4 in place, the extent to which one finds a proposition plausible may be quantified as follows:

Definition 5 (*Plausibility*). Given a BoE $\{\Theta, \mathscr{F}, m\}$, the *plausibility* of $A \subseteq \Theta$ is $\mathrm{Pl} : 2^\Theta \mapsto [0,1]$ where $\mathrm{Pl}(A) = 1 - \mathrm{Dou}(A) = 1 - \mathrm{Bel}(\bar{A})$.

Indeed $\mathrm{Pl}(A)$ indicates the extent to which one fails to doubt A, i.e., the extent to which one finds A to be plausible. One may easily show that, for any $A \subseteq \Theta$, $\mathrm{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(B)$ and $\mathrm{Pl}(A) \geqslant \mathrm{Bel}(A)$.

The *uncertainty* associated with A, denoted by $\mathrm{Un}(A)$, is the interval $\mathrm{Un}(A) = [\mathrm{Bel}(A), \mathrm{Pl}(A)]$. Note that $0 \leqslant \ell[\mathrm{Un}(A)] \leqslant \min\{1 - \mathrm{Bel}(A), \mathrm{Pl}(A)\}$.

When each focal set contains only one element, i.e., $m(A) = 0 \ \forall |A| \neq 1$, belief functions become probability functions. In such a case, it is easy to show the following:

(1) The BBA $m(A)$ reduces to probability, i.e., $m(A) = P(A)$.
(2) $\mathrm{Bel}(A) = \mathrm{Pl}(A) = P(A)$.
(3) $\mathrm{Bel}(A \cup B) = \mathrm{Bel}(A) + \mathrm{Bel}(B)$ whenever $A, B \subseteq \Theta$ and $A \cap B = \emptyset$.
(4) $\mathrm{Bel}(A) + \mathrm{Bel}(\bar{A}) = 1$.
(5) $\ell[\mathrm{Un}(A)] = 0$.

Dempster's rule of combination (DRC) can be used to find a new BBA (and correspondingly a new BoE) that combines and takes into account several BBAs (and correspondingly several BoEs) that span the *same* FoD.

Definition 6 (*Dempster's rule of combination* (*DRC*)). The orthogonal sum denoted by $m_1 \oplus m_2 : 2^\Theta \mapsto [0,1]$ of two BBAs $m_1 : 2^\Theta \mapsto [0,1]$ and $m_2 : 2^\Theta \mapsto [0,1]$ defined over the same FoD Θ is the following: for $\forall A \subseteq \Theta$

$$(m_1 \oplus m_2)(A) = \frac{\sum_{C,D:C\cap D=A} m_1(C)m_2(D)}{1 - \sum_{C,D:C\cap D=\emptyset} m_1(C)m_2(D)} \quad \forall C, D \subseteq \Theta, \tag{2}$$

if

$$\sum_{C\cap D=\emptyset} m_1(C)m_2(D) < 1. \tag{3}$$

Note that $m_1 \oplus m_2 : 2^\Theta \mapsto [0,1]$ is also a BBA in the sense of Definition 1. The pair of propositions $\{C, D\}$ is said to be *compatible* if $C \cap D \neq \emptyset$; otherwise they are said to be *incompatible* or *disjoint*. Two BBAs are said to be *compatible* if (3) is satisfied. DRC is applicable to such compatible BBAs only.

3. Conditioning evidence in DS theory

Various notions of conditional belief and plausibility have been reported previously [27–34]. The conditional measures in [27] are derived by relating notions of inner and outer measures to DS notions. As had been pointed out in

[27], same or similar expressions for these same conditional measures appear in previous other articles as well [1,28,35].

Theorem 7 [27]. *Given a BoE* $\{\Theta, \mathscr{F}, m\}$ *and* $A \in \widehat{\mathscr{F}}(\Theta)$, *the conditional belief* $\mathrm{Bel}(B|A) : 2^\Theta \mapsto [0,1]$ *and conditional plausibility* $\mathrm{Pl}(B|A) : 2^\Theta \mapsto [0,1]$ *assigned to* $B \subseteq \Theta$ *are*

$$\mathrm{Bel}(B|A) = \frac{\mathrm{Bel}(A\cap B)}{\mathrm{Bel}(A\cap B) + \mathrm{Pl}(A - B)},$$
$$\mathrm{Pl}(B|A) = \frac{\mathrm{Pl}(A\cap B)}{\mathrm{Pl}(A\cap B) + \mathrm{Bel}(A - B)}. \tag{4}$$

From now on, we use $m(\cdot|A) : 2^\Theta \mapsto [0,1]$ with $m(\emptyset|A) = 0$ to denote the *conditional BBA given A* corresponding to the conditional notions $\mathrm{Bel}(\cdot|A) : 2^\Theta \mapsto [0,1]$ and $\mathrm{Pl}(\cdot|A) : 2^\Theta \mapsto [0,1]$ in Theorem 7; its existence is in fact demonstrated in [27].

Note that, Theorem 7 implies the following:

$$\mathrm{Bel}(B|A) = \mathrm{Bel}(A\cap B|A) \quad \text{and} \quad \mathrm{Pl}(B|A) = \mathrm{Pl}(A\cap B|A), \tag{5}$$

i.e., conditioning of B with respect to A actually applies to the propositions that are in common to both A and B. In other words, in evaluating the evidence we have to support B when our view is restricted to only A, *it only makes sense to consider the propositions both A and B have access to*!

The uncertainty intervals can be used for an alternate approach to interpret conditional notions in the following manner as well: Consider a BoE $\{\Theta, \mathscr{F}, m\}$ and propositions A and B such that $B \subseteq \Theta$ and $A \in \widehat{\mathscr{F}}(\Theta)$. The uncertainty intervals one may associate with A and B with respect to the FoD Θ, viz., $\mathrm{Un}(A)$ and $\mathrm{Un}(B)$, respectively, do not account for the 'contents' of A and hence no measure of the 'contribution' of $\mathrm{Un}(A)$ towards $\mathrm{Un}(B)$ may be extracted. Such a conditional uncertainty measure is exactly what would be useful if the knowledge we have were to be restricted to what is available in A. Indeed, what is desired would be an appropriate uncertainty interval for B that uses *only* those focal elements of Θ that are used to compute $\mathrm{Bel}(A)$. The difficulty stems from the fact that the BBA associated with only these focal elements do not constitute a BBA in the sense of Definition 1. In other words, given $\mathrm{Un}(A)$, it is not possible to directly identify the contribution of $\mathrm{Un}(A)$ towards $\mathrm{Un}(B)$. The more appropriate notion is the *conditional uncertainty interval* $\mathrm{Un}(B|A) = [\mathrm{Bel}(B|A), \mathrm{Pl}(B|A)]$, $B \subseteq \Theta$, $A \in \widehat{\mathscr{F}}(\Theta)$.

At this juncture, we must mention that these conditional notions in Theorem 7 are not 'commutative' in the sense that conditioning first with respect to A_1 and subsequently with respect to A_2 is not in general equivalent to conditioning with respect to $A_1 \cap A_2$. We comment on this issue further in Section 6.

3.1. Conditional BBA

The results in this section shed light on how conditioning impacts the original BBA prior to conditioning. This viewpoint, which is based directly on the BBA, we believe is more intuitive in the sense that it enables one to discern how the original masses are impacted upon receiving the conditioning evidence.

First, we identify those propositions whose masses are annulled with conditioning.

Lemma 8. *Given the BoE* $\{\Theta, \mathscr{F}, m\}$ *and* $A \in \widehat{\mathscr{F}}(\Theta)$, *consider the conditional BBA* $m(\cdot|A) : 2^{\Theta} \mapsto [0,1]$. *Then* $m(B|A) = 0$ *whenever* $\bar{A} \cap B \neq \emptyset$.

Proof. Suppose $\exists B \subseteq \Theta$ s.t. $\bar{A} \cap B \neq \emptyset$ and $m(B|A) > 0$. Then we must have $\mathrm{Bel}(B|A) > \mathrm{Bel}(A \cap B|A)$, because the non-zero mass $m(B|A)$ 'contributes' towards $\mathrm{Bel}(B|A)$ but not towards $\mathrm{Bel}(A \cap B|A)$. But, this contradicts (5) which states that $\mathrm{Bel}(B|A) = \mathrm{Bel}(A \cap B|A)$. □

In other words, Lemma 8 states that, conditioning annuls masses of all those propositions that *may* imply the complement of the conditioning proposition. What happens to the masses of the remaining propositions, viz., those that *definitely* imply the conditioning proposition?

To address this, we proceed as follows:

$$\begin{aligned}
&\mathrm{Pl}(A) - \mathrm{Bel}(A \cap B) - \mathrm{Pl}(A - B) \\
&\quad = \mathrm{Pl}(A) - \mathrm{Bel}(A \cap B) - [1 - \mathrm{Bel}(\bar{A} \cup B)] \\
&\quad = \mathrm{Pl}(A) - \mathrm{Bel}(A \cap B) - [\mathrm{Pl}(A) + \mathrm{Bel}(\bar{A})] + \mathrm{Bel}(\bar{A} \cup B) \\
&\quad = \mathrm{Bel}(\bar{A} \cup B) - \mathrm{Bel}(\bar{A}) - \mathrm{Bel}(A \cap B),
\end{aligned} \tag{6}$$

where we have used the relationships $\mathrm{Pl}(A) + \mathrm{Bel}(\bar{A}) = 1$ and $\mathrm{Pl}(A - B) + \mathrm{Bel}(\bar{A} \cup B) = 1$. But, we realize that

$$\mathrm{Bel}(\bar{A} \cup B) - \mathrm{Bel}(\bar{A}) - \mathrm{Bel}(A \cap B) = \sum_{X : X \in \mathscr{S}(A \cap B)} m(X), \tag{7}$$

where

$$\mathscr{S}(A \cap B) \doteq \{X \in \mathscr{F}(\Theta) : X = D \cup C \text{ s.t. } \emptyset \neq D \subseteq \bar{A},\ \emptyset \neq C \subseteq A \cap B\}. \tag{8}$$

Hence we have

$$\mathrm{Pl}(A) - \sum_{X : X \in \mathscr{S}(A \cap B)} m(X) = \mathrm{Bel}(A \cap B) + \mathrm{Pl}(A - B). \tag{9}$$

Therefore we may express the conditional belief as

$$\mathrm{Bel}(B\,|\,A) = \frac{\mathrm{Bel}(A\cap B)}{\mathrm{Bel}(A\cap B)+\mathrm{Pl}(A-B)} = \frac{\mathrm{Bel}(A\cap B)}{\mathrm{Pl}(A)-\sum_{X:X\in\mathscr{S}(A\cap B)} m(X)}. \tag{10}$$

It is this expression that we intend to use to further study the conditional BBA. For convenience, from now on, with no loss of generality, we assume that $B\subseteq A$; after all, Lemma 8 already implies that $m(B\,|\,A)=0$ whenever $\bar{A}\cap B\neq\emptyset$. For this case, (10) reduces to

$$\mathrm{Bel}(B\,|\,A) = \frac{\mathrm{Bel}(B)}{\mathrm{Bel}(B)+\mathrm{Pl}(A-B)} = \frac{\mathrm{Bel}(B)}{\mathrm{Pl}(A)-\Xi(B)}, \tag{11}$$

where we use the short-hand notation

$$\Xi(B) = \sum_{X:X\in\mathscr{S}(B)} m(X). \tag{12}$$

Note that

$$\mathscr{S}(B) \doteq \{X\in\mathscr{F}(\Theta) : X = D\cup C \text{ s.t. } \emptyset\neq D\subseteq\bar{A}, \emptyset\neq C\subseteq B\subseteq A\}. \tag{13}$$

For the discussion to follow, it will also be convenient to define the following:

$$\hat{\mathscr{S}}(X) \doteq \{B\subseteq A : B\supseteq A\cap X \text{ where } X\in\mathscr{F}(\Theta) \text{ and } \emptyset\neq\bar{A}\cap X\}. \tag{14}$$

See Fig. 1.

Use Definition 2 in (11) to get

$$m(B\,|\,A) = \frac{1}{\mathrm{Pl}(A)-\sum_{X:X\in\mathscr{S}(B)} m(X)} \sum_{C:C\subseteq B} m(C) - \sum_{C:C\subset B} m(C\,|\,A). \tag{15}$$

Let us study (15) in more detail: observe that $m(B\,|\,A)$, for a given B, depends *only* on the following quantities:

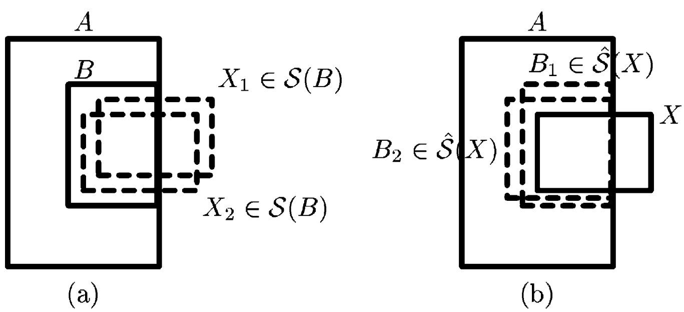

Fig. 1. (a) The set $\mathscr{S}(B)$, $B\subseteq A$ and (b) the set $\widehat{\mathscr{S}}(X)$, $X\in\mathscr{F}(\Theta)$, $\emptyset\neq\bar{A}\cap X$.

(1) $\mathrm{Pl}(A)$—plausibility of the conditioning proposition which is *independent* of the proposition being conditioned.
(2) $m(C)$, $C \subseteq B$—originally assigned masses of propositions that *definitely* imply the proposition being conditioned.
(3) $m(X)$, $X \in \mathscr{S}(B)$—originally assigned masses of propositions that *may* imply $\bar{A}$ *and* B but *definitely not* $A - B$.
(4) $m(C|A)$, $C \subset B$—conditional masses of propositions that *definitely* imply the proposition being conditioned; since $\mathscr{S}(C) \subseteq \mathscr{S}(B)$, $\forall C \subseteq B$, these conditional masses in turn depend only on $m(X)$, $X \in \mathscr{S}(B)$.
(5) Perhaps more important is to observe that $m(C)$, $C \subseteq \bar{A}$—masses of propositions that *definitely* imply $\bar{A}$—make no contribution towards *any* proposition being conditioned.

With the above development in place, is it possible to view the conditioning operation as an annulment and re-distribution of the masses of those propositions that *may* imply the complement of the conditioning proposition? If so, do these masses get re-distributed or not re-distributed (which will require a re-normalization)? How exactly does this re-distribution take place?

To address these questions, we now provide an explicit recursive formula that enables one to compute the conditional BBA. First, express (11) as

$$\sum_{C:C\subseteq B} \mathrm{Pl}(A)m(C|A) = \sum_{C:C\subseteq B} [m(C) + \Xi(B)m(C|A)]. \tag{16}$$

Subtract $\sum_{C:C\subseteq B} \Xi(C)m(C|A)$ from each side and re-arrange terms to get

$$\Delta(B) = \sum_{C:C\subset B} [\Xi(B) - \Xi(C)]m(C|A) - \sum_{C:C\subset B} \Delta(C), \tag{17}$$

where we use the notation

$$\Delta(B) = [\mathrm{Pl}(A) - \Xi(B)]m(B|A) - m(B). \tag{18}$$

Then we have

Lemma 9. *Given the BoE* $\{\Theta, \mathscr{F}, m\}$ *and* $A \in \widehat{\mathscr{F}}(\Theta)$, *consider the conditional BBA* $m(\cdot|A) : 2^{\Theta} \mapsto [0,1]$. *Then, for* $B \subseteq A$ *and every collection* $\{D_i\}$ *such that* $C \subseteq D \subset B$,

$$\begin{aligned}\Delta(B) &= \sum_{C:C\subset B} \left[\Xi(B) - \sum_{I\neq\emptyset;I\subseteq\{1,\dots,2^{|B|-|C|}-1\}} (-1)^{|I|+1}\Xi\left(\bigcap_{i\in I} D_i\right)\right] m(C|A) \\ &= \sum_{C:C\subset B} \left[\Xi(B) - \sum_{i=|C|}^{|B|-1}(-1)^{|B|-1-i} \sum_{D:C\subseteq D\subset B;|D|=i} \Xi(D)\right] m(C|A).\end{aligned} \tag{19}$$

Proof. The fact that the two alternate expressions in the right-hand side of (19) are identical is easy to see. We establish the claim via the second expression through induction on $|B| = \{1, 2, \ldots\}$.

(i) $|B| = 1$ *case*. In this case, (19) yields $\Delta(B) = 0$, which may be verified to be true via (17).

(ii) $|B| = \{1, 2, \ldots, M\}$ *cases*. Suppose the claim is true for $1 \leqslant |B| \leqslant M$.

(iii) $|B| = M + 1$ *case*. Consider (17). Note that the sets $C : C \subset B$ in the last term on the right-hand side of (17) satisfy the property $|C| \leqslant M$. Hence we may apply (19) to get

$$\begin{aligned}
&\sum_{C:C\subset B} \Delta(C) \\
&= \sum_{C:C\subset B} \sum_{D:D\subset C} \left[\Xi(C) - \sum_{i=|D|}^{|C|-1} (-1)^{|C|-1-i} \sum_{E:D\subseteq E\subset C;|E|=i} \Xi(E) \right] m(D|A) \\
&= \sum_{C:C\subset B} \sum_{D:C\subset D\subset B} \left[\Xi(D) - \sum_{i=|C|}^{|D|-1} (-1)^{|D|-1-i} \sum_{E:C\subseteq E\subset D;|E|=i} \Xi(E) \right] m(C|A).
\end{aligned} \tag{20}$$

Substitute in (17) to get

$$\begin{aligned}
\Delta(B) = &\sum_{C:C\subset B} \Xi(B) m(C|A) \\
&- \sum_{C:C\subset B} \sum_{D:C\subseteq D\subset B} \left[\Xi(D) - \sum_{i=|C|}^{|D|-1} (-1)^{|D|-1-i} \sum_{E:C\subseteq E\subset D;|E|=i} \Xi(E) \right] m(C|A).
\end{aligned} \tag{21}$$

Now, compare (19) and (21). Clearly, the claim will be established if we can show the following: for a given $B : B \subseteq A$ and $C : C \subset B$

$$\begin{aligned}
&\sum_{i=|C|}^{|B|-1} (-1)^{|B|-1-i} \sum_{D:C\subseteq D\subset B;|D|=i} \Xi(D) \\
&= \sum_{D:C\subseteq D\subset B} \left[\Xi(D) - \sum_{i=|C|}^{|D|-1} (-1)^{|D|-1-i} \sum_{E:C\subseteq E\subset D;|E|=i} \Xi(E) \right].
\end{aligned} \tag{22}$$

To verify this identity, consider the coefficient associated with the arbitrary term $\Xi(X)$, $X \subset B$:

(iii.a) Left-hand side of (22). Only one coefficient is generated; it corresponds to $i = |X|$. This yields $(-1)^{|B|-1-|X|}$.

(iii.b) Right-hand side of (22). Only one coefficient of value 1 is generated by the first term; to get the coefficients generated by the second term, we need to put $i = |X|$ and consider all sets $D : X \subset D \subset B$ ($D = X$ need not be considered since the second term vanishes in this situation). This yields

$$\begin{aligned}1 - \sum_{D:X\subset D\subset B} (-1)^{|D|-1-|X|} &= 1 - \sum_{i=|X|+1}^{|B|-1} \binom{|B|-|X|}{i-|X|} (-1)^{i-1-|X|} \\ &= 1 + \sum_{i=|X|+1}^{|B|-1} \binom{|B|-|X|}{i-|X|} (-1)^{i-|X|} \\ &= 1 + \sum_{i=1}^{|B|-1-|X|} \binom{|B|-|X|}{i} (-1)^{i} \\ &= 1 + \sum_{i=0}^{|B|-|X|} \binom{|B|-|X|}{i} (-1)^{i} \\ &\quad - \binom{|B|-|X|}{|B|-|X|} (-1)^{|B|-|X|} - \binom{|B|-|X|}{0} (-1)^{0} \\ &= -\binom{|B|-|X|}{|B|-|X|} (-1)^{|B|-|X|} = (-1)^{|B|-1-|X|}, \end{aligned} \tag{23}$$

where we used the fact that $\sum_{i=0}^{|B|-|X|} \binom{|B|-|X|}{i} (-1)^{i} = 0$.

Hence, (22) indeed holds true. This establishes the claim in (19). □

In effect, Lemma 9 demonstrates the fact that the conditional BBA of a proposition is completely determined by the conditional BBAs of those propositions that imply it. With this in place, it is easy to establish.

Lemma 10. *Given the BoE* $\{\Theta, \mathscr{F}, m\}$ *and* $A \in \widehat{\mathscr{F}}(\Theta)$, *consider the conditional BBA* $m(\cdot\,|A) : 2^{\Theta} \mapsto [0,1]$. *Then*

$$\frac{m(B)}{\mathrm{Pl}(A) - \sum_{X:X\in\mathscr{S}(B)} m(X)} \leqslant m(B\,|A) \leqslant \frac{\mathrm{Bel}(B)}{\mathrm{Pl}(A) - \sum_{X:X\in\mathscr{S}(B)} m(X)} \quad \forall B \subseteq A. \tag{24}$$

Proof. The right-hand side inequality simply claims that $m(B\,|A) \leqslant \mathrm{Bel}(B\,|A)$ (see (9)) which is of course obvious. The left-hand side inequality claims that $\Delta(B) \geqslant 0$. To show this, consider (19) and observe that

$$\bigcup_{i\in I} D_i = \begin{cases} B & \text{for } |C| \neq |B|-1, \\ C & \text{for } |C| = |B|-1. \end{cases} \tag{25}$$

Now noting the definition of $\Xi(B)$ in (12), it is clear that $\Xi(\cdot)$ satisfies the following property of belief functions (although it is not necessarily a belief function):

$$\Xi(B) \geqslant \sum_{I} (-1)^{|I|+1} \Xi\left(\bigcap_{i \in I} D_i \right), \quad \text{for } \emptyset \neq I \subseteq \{1, \ldots, 2^{|B|-|C|} - 1\}. \tag{26}$$

Thus $\Delta(B) \geqslant 0$, as claimed. □

Lemma 10 implies that

$$m(B) \leqslant \frac{m(B)}{\text{Pl}(A) - \sum_{X:X \in \mathscr{S}(B)} m(X)} \leqslant m(B|A) \quad \forall B \subseteq A, \tag{27}$$

i.e., the masses of those propositions that *definitely* imply the conditioning proposition cannot decrease with conditioning.

In summary, Lemmas 8 and 10 enable us to conclude that, conditioning annuls the masses of all propositions that *may* imply the complement of the conditioning proposition while increasing or keeping unchanged the masses of all propositions that *definitely* imply the conditioning proposition. Of this latter class of propositions, we can actually identify those that are guaranteed to increase after conditioning:

Lemma 11. *Given the BoE* $\{\Theta, \mathscr{F}, m\}$ *and* $A \in \widehat{\mathscr{F}}(\Theta)$, *consider the conditional BBA* $m(\cdot|A) : 2^{\Theta} \mapsto [0,1]$. *Then* $m(B) < m(B|A) \quad \forall B \subseteq A, \quad B \in \mathscr{F}(\Theta)$, *if* $\max\left\{ \text{Bel}(\bar{A}), \sum_{X:X \in \mathscr{S}(B)} m(X) \right\} > 0$.

Proof. From (15) it is clear that $m(B) < m(B|A)$, $\forall B \subseteq A$, if $m(B) < \frac{m(B)}{\text{Pl}(A) - \sum_{X:X \in \mathscr{S}(B)} m(X)}$. Since $B \in \mathscr{F}(\Theta)$, we have $m(B) > 0$. Hence, this is equivalent to

$$\text{Pl}(A) - \sum_{X:X \in \mathscr{S}(B)} m(X) < 1 \iff \sum_{X:X \in \mathscr{S}(B)} m(X) \begin{cases} \geqslant 0, & \text{if } \text{Pl}(A) < 1, \\ > 0, & \text{if } \text{Pl}(A) = 1. \end{cases} \tag{28}$$

The claim then follows. □

We also have

Lemma 12. *Given the BoE* $\{\Theta, \mathscr{F}, m\}$ *and* $A \in \widehat{\mathscr{F}}(\Theta)$, *consider the conditional BBA* $m(\cdot|A) : 2^{\Theta} \mapsto [0,1]$. *Then the following are true*:

(i)

$$\frac{m(A)}{\mathrm{Bel}(A)} \leqslant m(A\,|\,A) \leqslant 1. \tag{29}$$

(ii) *For all* $B \subseteq A$ *s.t.* $m(B) = \mathrm{Bel}(B)$

$$m(B\,|\,A) = \frac{m(B)}{\mathrm{Pl}(A) - \sum_{X:X\in\mathscr{S}(B)} m(X)} = \frac{m(B)}{m(B) + \mathrm{Pl}(A-B)}. \tag{30}$$

(iii) *For all* $B \subseteq A$ *s.t.* $B \in \mathscr{T}(\bar{A})$

$$m(B\,|\,A) = \frac{m(B)}{\mathrm{Pl}(A)}, \tag{31}$$

where $\mathscr{T}(\bar{A}) \doteq \{B \subseteq A \subseteq \Theta : \mathscr{S}(B) = \emptyset\}$.

Proof

(i) It is clear that $\mathrm{Pl}(A) - \sum_{X:X\in\mathscr{S}(A)} m(X) = \mathrm{Bel}(A)$. Now the claim follows by direct application of Lemma 10.

(ii) When $m(B) = \mathrm{Bel}(B)$, the upper and lower bounds in Lemma 10 converge; the claim then follows directly.

(iii) Note that (11) implies that $\mathrm{Bel}(B\,|\,A) = \frac{\mathrm{Bel}(B)}{\mathrm{Pl}(A)}\ \forall B \in \mathscr{T}(\bar{A})$. The claim then follows when one notices that $B \in \mathscr{T}(\bar{A}) \Rightarrow C \in \mathscr{T}(\bar{A})\ \forall C \subseteq B$. □

At this juncture, we wish to make several observations.

(1) Lemma 10 implies that the conditional mass of *all* propositions in $B \subseteq A$ must strictly increase whenever $\mathrm{Bel}(\bar{A}) > 0 \Longleftrightarrow \mathrm{Pl}(A) < 1$.

(2) Item (i) of Lemma 12 implies that the conditional mass of the conditioning proposition cannot decrease. When $\mathrm{Bel}(A) < 1$, $m(A\,|\,A)$ must necessarily exceed $m(A)$. When $m(A) = \mathrm{Bel}(A)$, $m(A\,|\,A) = 1$ and the mass of every other proposition is zero; a special case for which this is applicable is when A is a singleton proposition.

(3) Special cases for which item (ii) of Lemma 12 are applicable are the following:
   (a) When B is a singleton proposition; and/or
   (b) $\mathrm{Bel}(B) = 0$ (which implies $m(B) = 0$)—this situation actually yields $m(B\,|\,A) = 0, \forall B \subseteq A$ s.t. $\mathrm{Bel}(B) = 0$, i.e., the mass of a proposition having zero belief remains at zero with conditioning.

(4) Item (iii) of Lemma 12 exposes an important fact: one may view each element in $\mathscr{T}(\bar{A})$ as a proposition whose mass is not further 'refined' (except perhaps due to re-normalization by $\mathrm{Pl}(A)$) with conditioning. It is only the masses of those propositions that render a non-empty $\mathscr{S}(B)$ that may get further refined with conditioning.

3.1.1. Summary: BBA-based interpretation of conditioning

Summarizing the results in Lemmas 8–12, we may now interpret the conditioning operation as part annulment with re-distribution and part annulment without re-distribution of the originally allocated masses of propositions that may *imply* the complement of the conditioning proposition, viz., $X \subseteq \Theta$ s.t. $\bar{A} \cap X \neq \emptyset$

(1) *Masses that are annulled without re-distribution*. Masses of propositions $X \subseteq \Theta$ s.t. $X \subseteq \bar{A}$ are annulled but not re-distributed; this generates the re-normalization factor $\mathrm{Pl}(A)$. See Fig. 2(a).
(2) *Masses that are annulled with re-distribution*. Masses of propositions $X \subseteq \Theta$ s.t. $A \cap X \neq \emptyset$ *and* $\bar{A} \cap X \neq \emptyset$ are annulled but re-distributed toward $B \in \hat{\mathscr{S}}(X)$. See Fig. 2(b).
(3) *Masses that cannot decrease*. Masses of the remaining propositions (i.e., propositions B s.t. $B \subseteq A$) cannot decrease with conditioning. No proposition in $T(\bar{A})$ gets further refined (except due to the re-normalization factor $\mathrm{Pl}(A)$ in (2)). The remaining masses of propositions $B \subseteq A$ may benefit from the re-distribution in (2). In particular, only $m(X)$ $\forall X \in \mathscr{S}(B)$, can contribute towards $m(B|A)$, $B \subseteq A$. See Fig. 2(c).

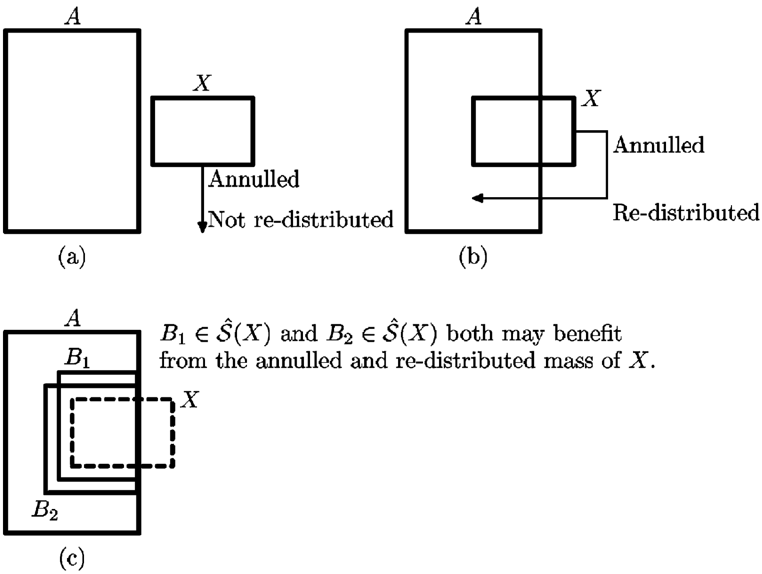

Fig. 2. Impact of conditioning on original mass assignment.

Example 13. It is perhaps best to illustrate the above notions via an example. Consider the situation in Table 1 where $\Theta = \{a, b, c, d\}$ and the conditioning proposition is $A = \{a, b, c\} \Rightarrow \bar{A} = \{d\}$. The deductions one may make regarding the conditional BBA are indicated as well. Note that

$$\mathscr{S}(\{a\}) = \mathscr{S}(\{a, b\}) = \{a, d\}; \quad \mathscr{S}(\{a, c\}) = \mathscr{S}(A) = \{\{a, d\}, \{c, d\}\}; \\ \mathscr{S}(\{b, c\}) = \{c, d\} \tag{32}$$

and

$$\hat{\mathscr{S}}(\{a, d\}) = \{a, \{a, b\}, \{a, c\}, \{a, b, c\}\}, \\ \hat{\mathscr{S}}(\{c, d\}) = \{c, \{a, c\}, \{b, c\}, \{a, b, c\}\}. \tag{33}$$

Table 2 shows the computed conditional notions.

3.2. Conditional belief and plausibility

We have already obtained several important results regarding how conditioning affects the original BBA. In this section, we make several observations regarding the conditional belief and plausibility notions. These provide further insight into the conditioning operation and will be useful in the development of the updating strategy proposed in Section 4.

Table 1
Illustrative example—originally cast BBA. $\Theta = \{a, b, c, d\}$ and $A = \{a, b, c\}$

B	Bel(B)	Pl(B)	$m(B)$	Deductions regarding $m(B \mid A)$
a	0.2	0.5	0.2	$\text{Pl}(A) < 1 \Rightarrow m(\{a\} \mid A) > m(\{a\})$ and $\mathscr{S}(\{a\})$ contributes
b	0.2	0.5	0.2	$\{b\} \in \mathscr{T}(\bar{A}) \Rightarrow m(\{b\} \mid A) = m(\{b\})/\text{Pl}(A)$
c	0	0.2	0	$m(\{c\}) = \text{Bel}(\{c\}) = 0 \Rightarrow m(\{c\} \mid A) = 0$
d	0.1	0.3	0.1	$\{d\} \subseteq \bar{A} \Rightarrow m(\{d\} \mid A) = 0$ and mass is not re-distributed
$\{a, b\}$	0.6	0.8	0.2	$\text{Pl}(A) < 1 \Rightarrow m(\{a, b\} \mid A) > m(\{a, b\})$ and $\mathscr{S}(\{a, b\})$ contributes
$\{a, c\}$	0.2	0.7	0	$\text{Pl}(A) < 1 \Rightarrow m(\{a, c\} \mid A) > m(\{a, c\})$ and $\mathscr{S}(\{a, c\})$ contributes
$\{a, d\}$	0.4	0.7	0.1	$\bar{A} \cap \{a, d\} \neq \emptyset \Rightarrow m(\{a, d\} \mid A) = 0$ and mass is re-distributed to $\hat{\mathscr{S}}(\{a, d\})$
$\{b, c\}$	0.3	0.6	0.1	$\text{Pl}(A) < 1 \Rightarrow m(\{b, c\} \mid A) > m(\{b, c\})$ and $\mathscr{S}(\{b, c\})$ contributes
$\{c, d\}$	0.2	0.4	0.1	$\bar{A} \cap \{c, d\} \neq \emptyset \Rightarrow m(\{c, d\} \mid A) = 0$ and mass is re-distributed to $\hat{\mathscr{S}}(\{c, d\})$
$\{a, b, c\}$	0.7	0.9	0	$\text{Pl}(A) < 1 \Rightarrow m(\{A\} \mid A) > m(\{A\})$ and $\mathscr{S}(A)$ contributes

Table 2
Illustrative example—conditional BBA

B	$\mathrm{Bel}(B\,\vert\,A)$	$\mathrm{Pl}(B\,\vert\,A)$	$m(B\,\vert\,A)$
a	$\frac{0.2}{0.8}$	$\frac{0.5}{0.8}$	$\frac{0.2}{0.8}$
b	$\frac{0.2}{0.9}$	$\frac{0.5}{0.7}$	$\frac{0.2}{0.9}$
c	0	$\frac{0.2}{0.5}$	0
d	0	0	0
$\{a,b\}$	$\frac{0.6}{0.8}$	1.0	$\frac{0.25}{0.9}$
$\{a,c\}$	$\frac{0.2}{0.7}$	$\frac{0.7}{0.9}$	$\frac{0.02}{(0.7)(0.8)}$
$\{a,d\}$	$\frac{0.2}{0.8}$	$\frac{0.5}{0.8}$	0
$\{b,c\}$	$\frac{0.3}{0.8}$	$\frac{0.6}{0.8}$	$\frac{0.11}{(0.8)(0.9)}$
$\{c,d\}$	0	$\frac{0.2}{0.5}$	0
$\{a,b,c\}$	1	1	$\frac{0.031}{(0.7)(0.8)(0.9)}$

$\Theta = \{a,b,c,d\}$ and $A = \{a,b,c\}$.

3.2.1. *Trivial cases*

It is easy to see that $\mathrm{Bel}(\emptyset\,|\,A) = \mathrm{Pl}(\emptyset\,|\,A) = 0$ and $\mathrm{Bel}(\Theta\,|\,A) = \mathrm{Pl}(\Theta\,|\,A) = 1$.

3.2.2. *Monotonicity*

Since belief and plausibility functions are monotone with respect to set inclusion, for $B_1 \subseteq B_2$, $\mathrm{Bel}(B_1\,|\,A) \leqslant \mathrm{Bel}(B_2\,|\,A)$ and $\mathrm{Pl}(B_1\,|\,A) \leqslant \mathrm{Pl}(B_2\,|\,A)$.

3.2.3. *$A = \Theta$ case*

In this case, $\mathrm{Bel}(B\,|\,A) = \mathrm{Bel}(B)$ and $\mathrm{Pl}(B\,|\,A) = \mathrm{Pl}(B)$.

3.2.4. *$B = A$ case*

In this case, $\mathrm{Bel}(A\,|\,A) = \mathrm{Pl}(A\,|\,A) = 1$.

3.2.5. *$A \subseteq B$, $B \neq \emptyset$, case*

In this case, $\mathrm{Bel}(B\,|\,A) = \mathrm{Pl}(B\,|\,A) = 1$.

3.2.6. *Probabilistic BBA*

In a probability framework when $m(\cdot) = \mathrm{Bel}(\cdot) = \mathrm{Pl}(\cdot) \doteq P(\cdot)$, we have

$$\mathrm{Bel}(B|A) = \mathrm{Pl}(B|A) = \frac{P(A\cap B)}{P(A\cap B)+P(A-B)} = \frac{P(A\cap B)}{P(A)} = P(B|A). \tag{34}$$

These observations illustrate that the notions in Theorem 7 can be considered 'natural extensions' [36], and hence act as generalizations, of those in Definitions 2 and 5.

With the above discussion in mind, these conditional notions can be used as measures to indicate the support provided by proposition A for another proposition B; or, to be more precise, for the propositions in common to both A and B, viz., $A\cap B$. Unlike the direct calculation of the belief using the complete BoE, these measures explicitly depend on the specific propositions in A that condition the propositions in B.

4. Updating evidence

Now that we have quantified how a given proposition A contributes to another proposition B (actually to $A\cap B$), how should we update our originally assigned support for B? In [37], perhaps for the first time, this updated belief of B conditional to A—which we denote by $\mathrm{Bel}_A(B)$—is taken to be a linear combination of the originally assigned belief $\mathrm{Bel}(B)$ and the conditional belief $\mathrm{Bel}(B|A)$. This strategy is simple, works directly on belief functions instead of the BBA [23] and accounts for both the individual evidence cast on B and the evidence gathered from what are common to both A and B in a unified manner. Moreover, as we will presently demonstrate, it allows one to accommodate the integrity and inertia of the available evidence and its flexibility to updating; it also possesses most of the properties that one expects from a reasonable updating strategy.

4.1. Updating strategy

Consider the following linear combination of $m(B)$ and $m(B|A)$ for updating the mass of B:

$$m_A(B) = \alpha_A m(B) + \beta_A m(B|A), \tag{35}$$

where $\{\alpha_A, \beta_A\}$ are parameters dependent on the conditioning proposition A. We use the subscript A to distinguish quantities that have been updated conditional to A. Note that $m_A(\emptyset) = 0$ while

$$\sum_{B:B\subseteq\Theta} m_A(B) = 1 \iff \alpha_A + \beta_A = 1. \tag{36}$$

The corresponding updated belief is then

$$\mathrm{Bel}_A(B) = \alpha_A \mathrm{Bel}(B) + \beta_A \mathrm{Bel}(B|A). \tag{37}$$

Note that $\mathrm{Bel}_A(\emptyset) = 0$ and $\mathrm{Bel}_A(\Theta) = 1$; the validity of item (iii) of Theorem 3 for $\mathrm{Bel}_A(\cdot)$ follows from the fact that the convex sum of belief functions constitutes another belief function.

Finally, the updated plausibility $\mathrm{Pl}_A(B)$ may be obtained via

$$\begin{aligned}\mathrm{Pl}_A(B) = 1 - \mathrm{Bel}_A(\overline{B}) &= 1 - [\alpha_A \mathrm{Bel}(\overline{B}) + \beta_A \mathrm{Bel}(\overline{B}|A)] \\ &= \alpha_A[1 - \mathrm{Bel}(\overline{B})] + \beta_A[1 - \mathrm{Bel}(\overline{B}|A)] \\ &= \alpha_A \mathrm{Pl}(B) + \beta_A \mathrm{Pl}(B|A),\end{aligned} \tag{38}$$

where we have used (36).

With the above development in place, we propose

Definition 14 (*Updated BBA, belief and plausibility*). Consider the BoE $\{\Theta, \mathscr{F}, m\}$ and a given $A \subseteq \widehat{\mathscr{F}}(\Theta)$. Then, for an arbitrary $B \subseteq \Theta$, define the following:

(i) *Updated BBA of B given A* is $m_A(B) : 2^\Theta \mapsto [0.1]$ where

$$m_A(B) = \alpha_A m(B) + \beta_A m(B|A). \tag{39}$$

(ii) The corresponding *updated belief of B given A* is $\mathrm{Bel}_A : 2^\Theta \mapsto [0,1]$ where

$$\mathrm{Bel}_A(B) = \alpha_A \mathrm{Bel}(B) + \beta_A \mathrm{Bel}(B|A). \tag{40}$$

(iii) The corresponding *updated plausibility of B given A* is $\mathrm{Pl}_A : 2^\Theta \mapsto [0,1]$ where

$$\mathrm{Pl}_A(B) = 1 - \mathrm{Bel}_A(\overline{B}) = \alpha_A \mathrm{Pl}(B) + \beta_A \mathrm{Pl}(B|A). \tag{41}$$

Here $\{\alpha_A, \beta_A\}$ are non-negative parameters dependent on the conditioning proposition A such that $\alpha_A + \beta_A = 1$.

For $B \subseteq \Theta$ and $A \in \widehat{\mathscr{F}}(\Theta)$, we may also define a corresponding *updated uncertainty interval* as [37]

$$\mathrm{Un}_A(B) = [\mathrm{Bel}_A(B), \mathrm{Pl}_A(B)] = \alpha_A \mathrm{Un}(B) + \beta_A \mathrm{Un}(B|A). \tag{42}$$

4.2. Properties of the updating strategy

We now discuss some of the properties of the updating strategy in Definition 14.

4.2.1. Updated BBA

From Lemmas 8–12, we observe the following:

$$m_A(B)\begin{cases} = \alpha_A m(B) \leqslant m(B), & \text{for } B \subseteq \Theta \text{ s.t. } \bar{A} \cap B \neq \emptyset; \\ \geqslant \left[\alpha_A + \dfrac{\beta_A}{\mathrm{Pl}(A) - \sum_{X:X\in\mathscr{S}(B)} m(X)}\right] m(B) \geqslant m(B), & \text{for } B \subseteq A. \end{cases} \tag{43}$$

For those propositions in A that are not refined from conditioning, we may be more precise:

$$m_A(B) = \left[\alpha_A + \frac{\beta_A}{\mathrm{Pl}(A)}\right] m(B) \geqslant m(B), \quad \forall B \subseteq A \text{ s.t. } B \in \mathscr{T}(\bar{A}). \tag{44}$$

In other words, the updating strategy in Definition 14 affects the originally assigned masses as follows:

(i) Masses of propositions that *may* imply the complement of the conditioning proposition are decreased (unless $\alpha_A = 1$).
(ii) Masses of propositions that *definitely* imply the conditioning proposition are increased (or at least not decreased).

Eqs. (43) and (44) bring to light the following important observation as well: propositions that do not allow further refinement in their mass when being conditioned do not get updated (except the changes due to re-normalization) either. In fact, we notice that

$$m_A(B) = m(B) \quad \forall B \subseteq A \text{ s.t. } B \in \mathscr{T}(\bar{A}), \quad \text{whenever } \mathrm{Pl}(A) = 1. \tag{45}$$

Clearly, these constitute important intuitively appealing properties that one expects from a reasonable updating strategy.

4.2.2. Updated belief and plausibility

4.2.2.1. Trivial cases. It is easy to see that $\mathrm{Bel}_A(\emptyset) = \mathrm{Pl}_A(\emptyset) = 0$ and $\mathrm{Bel}_A(\Theta) = \mathrm{Pl}_A(\Theta) = 1$.

4.2.2.2. Monotonicity. The fact that $\mathrm{Bel}(B)$ and $\mathrm{Bel}(B|A)$ each monotonically increases with respect to B implies that the same is true with $\mathrm{Bel}_A(B)$; $\mathrm{Pl}_A(B)$ possesses the same property as well.

4.2.2.3. $A = \Theta$ case. In this case, $\mathrm{Bel}_\Theta(B) = \mathrm{Bel}(B)$ and $\mathrm{Pl}_\Theta(B) = \mathrm{Pl}(B)$.

4.2.2.4. $B = A$ case. Note that $\mathrm{Bel}_A(A) = 1 - \alpha_A[1 - \mathrm{Bel}(A)]$ and $\mathrm{Pl}_A(A) = 1 - \alpha_A[1 - \mathrm{Pl}(A)]$. Now it is easy to see that $\mathrm{Bel}_A(A) \geqslant \mathrm{Bel}(A)$ and $\mathrm{Pl}_A(A) \geqslant \mathrm{Pl}(A)$, i.e., the occurrence of A improves its own belief and plausibility assignments.

We also get

$$\alpha_A = \begin{cases} \dfrac{1 - \mathrm{Bel}_A(A)}{1 - \mathrm{Bel}(A)} = \dfrac{1 - \mathrm{Pl}_A(A)}{1 - \mathrm{Pl}(A)} & \text{for } \mathrm{Bel}(A) \leqslant \mathrm{Pl}(A) < 1, \\ \dfrac{1 - \mathrm{Bel}_A(A)}{1 - \mathrm{Bel}(A)} & \text{for } \mathrm{Bel}(A) < \mathrm{Pl}(A) = 1, \\ \text{arbitrary in } [0, 1], & \text{for } \mathrm{Bel}(A) = \mathrm{Pl}(A) = 1. \end{cases} \tag{46}$$

4.2.2.5. $A \subseteq B, B \neq \emptyset$, case. In this case, $\mathrm{Bel}_A(B) = \alpha_A \mathrm{Bel}(B) + \beta_A \geqslant \mathrm{Bel}(B)$ and $\mathrm{Pl}_A(B) = \alpha_A \mathrm{Pl}(B) + \beta_A \geqslant \mathrm{Pl}(B)$.

4.2.2.6. Updated conditional. Another very intuitively appealing conclusion may be drawn as follows: from (40) and (41), note that

$$\begin{aligned} \mathrm{Bel}_A(A \cap B) &= \alpha_A \mathrm{Bel}(A \cap B) + \beta_A \mathrm{Bel}(B|A), \\ \mathrm{Pl}_A(A - B) &= \alpha_A \mathrm{Pl}(A - B) + \beta_A \mathrm{Pl}(A - B|A) \\ &= \alpha_A \mathrm{Pl}(A - B) + \beta_A \mathrm{Pl}(\overline{B}|A). \end{aligned} \tag{47}$$

Hence

$$\begin{aligned} \mathrm{Bel}_A(B|A) &= \frac{\mathrm{Bel}_A(A \cap B)}{\mathrm{Bel}_A(A \cap B) + \mathrm{Pl}_A(A - B)} \\ &= \frac{\alpha_A \mathrm{Bel}(A \cap B) + \beta_A \mathrm{Bel}(B|A)}{\alpha_A[\mathrm{Bel}(A \cap B) + \mathrm{Pl}(A - B)] + \beta_A[\mathrm{Bel}(B|A) + \mathrm{Pl}(\overline{B}|A)]} \\ &= \frac{\alpha_A \mathrm{Bel}(A \cap B) + \beta_A \mathrm{Bel}(B|A)}{\alpha_A[\mathrm{Bel}(A \cap B) + \mathrm{Pl}(A - B)] + \beta_A} = \mathrm{Bel}(B|A), \end{aligned} \tag{48}$$

i.e., $\mathrm{Bel}(B|A)$ is invariant with updating.

4.2.2.7. Repeated conditioning. Let us use the superscript (i) to denote the i times repeated conditioning with respect to proposition A via repeated application of (40), i.e.

$$\mathrm{Bel}_A^{(i+1)}(B) = \alpha_A^{(i)} \mathrm{Bel}_A^{(i)}(B) + \beta_A^{(i)} \mathrm{Bel}_A^{(i)}(B|A) \quad \forall i \geqslant 0, \tag{49}$$

where $i = 0$ and $i = 1$ denote terms related to the originally cast BBA and the first update respectively. Then

$$\begin{aligned} \mathrm{Bel}_A^{(2)}(B) &= \alpha_A^{(1)}[\alpha_A^{(0)} \mathrm{Bel}(B) + \beta_A^{(0)} \mathrm{Bel}(B|A)] + \beta_A^{(1)} \mathrm{Bel}_A^{(1)}(B|A) \\ &= \mathrm{Bel}(B|A) + \alpha_A^{(0)} \alpha_A^{(1)}[\mathrm{Bel}(B) - \mathrm{Bel}(B|A)], \end{aligned} \tag{50}$$

where we have used the invariance of $\mathrm{Bel}(B\,|\,A)$ in (48) and $\alpha_A^{(i)} + \beta_A^{(i)} = 1\ \forall i \geqslant 0$. Continuing in this manner, one may show that

$$\mathrm{Bel}_A^{(n)}(B) = \mathrm{Bel}(B\,|\,A) + \prod_{i=0}^{n-1} \alpha_A^{(i)}[\mathrm{Bel}(B) - \mathrm{Bel}(B\,|\,A)] \quad \forall n \geqslant 0, \tag{51}$$

where $\{\alpha_A^{(0)}, \beta_A^{(0)}\} \doteq \{\alpha_A, \beta_A\}$. This again results in the intuitively appealing result

$$\lim_{n\to\infty} \mathrm{Bel}_A^{(n)}(B) = \mathrm{Bel}(B\,|\,A), \quad \lim_{n\to\infty} \mathrm{Bel}_A^{(n)}(A) = 1. \tag{52}$$

4.2.2.8. Updated incremental. We also have

$$\begin{aligned} \mathrm{Bel}_A(B) - \mathrm{Bel}_A(A \cap B) &= \alpha_A[\mathrm{Bel}(B) - \mathrm{Bel}(A \cap B)], \\ \mathrm{Pl}_A(B) - \mathrm{Pl}_A(A \cap B) &= \alpha_A[\mathrm{Pl}(B) - \mathrm{Pl}(A \cap B)]. \end{aligned} \tag{53}$$

4.2.2.9. Behavior of updates. When is the updated notion higher than its corresponding original assignment? This is of course true whenever $B \subseteq A$ (as can be inferred from (43). Another way to address this question is to note that

$$\begin{aligned} \mathrm{Bel}_A(B) - \mathrm{Bel}(B) &= (1 - \alpha_A)[\mathrm{Bel}(B\,|\,A) - \mathrm{Bel}(B)], \\ \mathrm{Pl}_A(B) - \mathrm{Pl}(B) &= (1 - \alpha_A)[\mathrm{Pl}(B\,|\,A) - \mathrm{Pl}(B)], \\ \mathrm{Un}_A(B) - \mathrm{Un}(B) &= (1 - \alpha_A)[\mathrm{Un}(B\,|\,A) - \mathrm{Un}(B)]. \end{aligned} \tag{54}$$

The dynamics of these incrementals of belief, plausibility and uncertainty interval are therefore identical. It is now easy to arrive at the following interesting conclusions:

(1) $\mathrm{Bel}(B) \geqslant \mathrm{Bel}(B\,|\,A)$ and $\mathrm{Pl}(B) \geqslant \mathrm{Pl}(B\,|\,A)$ guarantee no increase in the updates for the belief and plausibility functions respectively; the minimum updates are limited by $\mathrm{Bel}(B\,|\,A)$ and $\mathrm{Pl}(B\,|\,A)$ respectively. Moreover, $\mathrm{Un}(B) \geqslant \mathrm{Un}(B\,|\,A)$ guarantees no deterioration of the uncertainty interval; the maximum improvement is limited by $\mathrm{Un}(B\,|\,A)$.
(2) $\mathrm{Bel}(B) \leqslant \mathrm{Bel}(B\,|\,A)$ and $\mathrm{Pl}(B) \leqslant \mathrm{Pl}(B\,|\,A)$ guarantee no decrease in the updates for the belief and plausibility functions respectively; the maximum updates are limited by $\mathrm{Bel}(B\,|\,A)$ and $\mathrm{Pl}(B\,|\,A)$ respectively. Moreover, $\mathrm{Un}(B) \leqslant \mathrm{Un}(B\,|\,A)$ guarantees no improvement of the uncertainty interval; the maximum deterioration is limited by $\mathrm{Un}(B\,|\,A)$.

These observations are summarized in Fig. 3.

4.3. Linear combination weights

In this section, we propose several strategies that enable the selection of the linear combination weights $\{\alpha_A, \beta_A\}$.

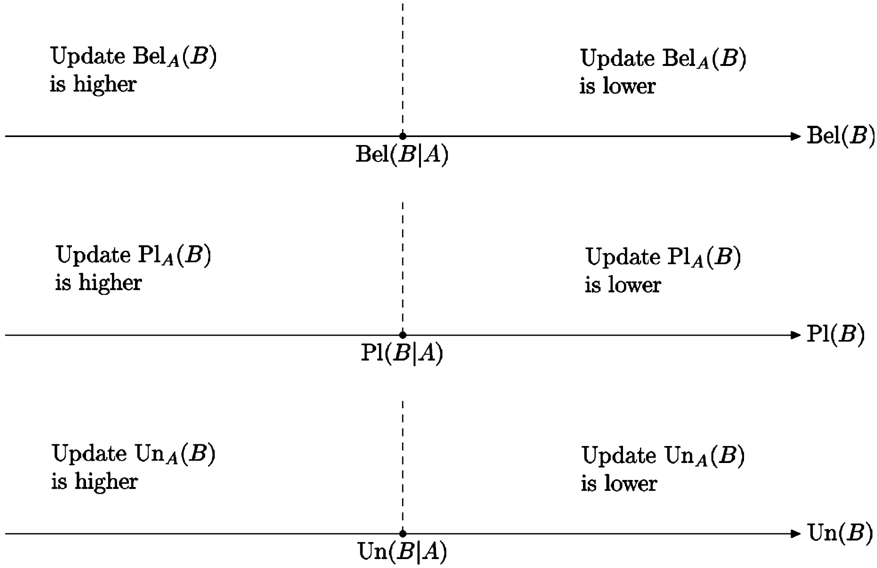

Fig. 3. Behavior of the updates $\mathrm{Bel}_A(B)$, $\mathrm{Pl}_A(B)$ and $\mathrm{Un}_A(B)$.

4.3.1. Inertia of available evidence

The weights $\{\alpha_A, \beta_A\}$ can be interpreted as measures that indicate the flexibility or inertia of the originally cast evidence to updating when presented with the incoming conditional proposition.

Definition 15 (*Inertia of available evidence based updating*). Consider the evidence updating strategy in Definition 14.

(i) The choice $\{\alpha_A, \beta_A\} = \{1, 0\}$ is called the *infinite inertia based* (*II-based*) *updating strategy*.
(ii) The choice $\{\alpha_A, \beta_A\} = \{0, 1\}$ is called the *zero inertia based* (*ZI-based*) *updating strategy*.
(iii) The choice $\{\alpha_A, \beta_A\} = \{\frac{N}{N+1}, \frac{1}{N+1}\}$, where N refers to the number of 'pieces' of evidence the available evidence is based upon is called the *proportional inertia based* (PI-based) *updating strategy*.

We make several observations regarding these updating strategies:

(1) *II-based updating*. This can account for the complete inflexibility of the available evidence towards changes (e.g., when it perceives the incoming evidence to be completely unreliable, when the original BoE is formed from a vast collection of reliable data thus generating a high inertia, etc.).

(2) *ZI-based updating*. This can account for the complete flexibility of the available evidence towards changes (e.g., when it perceives the incoming evidence to be completely reliable, when the original BoE has little or no credible knowledge base to begin with, etc.).
(3) *PI-based updating*. This strategy treats each ‘piece’ of already gathered evidence and the new ‘piece’ of incoming evidence as having equal inertia.

4.3.2. Integrity of available evidence

Suppose that the conditioning proposition A has just occurred, and we are in the process of updating the support for *all* the propositions (*including A* itself) armed with this new evidence. Then, for the integrity of the originally cast evidence to be maintained, it is reasonable to enforce $\mathrm{Bel}_A(A) \leqslant \mathrm{Pl}(A)$. The corresponding weights $\{\alpha_A, \beta_A\}$ we believe can be considered the most reasonable choices if we are unwilling to compromise the evidence that had already been cast prior to the arrival of the new evidence. In other words, in order not to contradict the originally cast evidence, we allow $\mathrm{Bel}_A(A)$ to increase to $\mathrm{Pl}(A)$—but no more! Substituting this in (40) we get

Definition 16 (*Integrity of available evidence based updating*). Consider the evidence updating strategy in Definition 14. The *integrity of available evidence based* (IAE-based) *updating strategy* refers to

$$\alpha_A \in \begin{cases} \left[\dfrac{1-\mathrm{Pl}(A)}{1-\mathrm{Bel}(A)}, 1\right] & \text{for } \mathrm{Bel}(A) < 1, \\ [0,1] & \text{for } \mathrm{Bel}(A) = \mathrm{Pl}(A) = 1. \end{cases} \tag{55}$$

We make several observations regarding this IAE-based updating strategy:

(1) α_A achieves its upper bound, i.e., $\alpha_A = 1$. This means that the current knowledge base is not changed. It indicates that the BoE is least flexible to the incoming evidence, viz., the II-based updating strategy.
(2) α_A achieves its lower bound, i.e., $\alpha_A = \frac{1-\mathrm{Pl}(A)}{1-\mathrm{Bel}(A)}$. This yields

$$\begin{aligned} &\mathrm{Bel}_A(A) = \mathrm{Pl}(A), \\ &\mathrm{Pl}_A(A) = \begin{cases} \dfrac{[\mathrm{Pl}(A) - \mathrm{Bel}(A)] + \mathrm{Pl}(A)[1-\mathrm{Pl}(A)]}{1-\mathrm{Bel}(A)} & \text{for } \mathrm{Bel}(A) < 1, \\ 1 & \text{for } \mathrm{Bel}(A) = \mathrm{Pl}(A) = 1. \end{cases} \end{aligned} \tag{56}$$

It indicates that the BoE is most flexible to the incoming evidence *to the extent that its own evidence is not compromised*. We refer to this as the *most flexible IAE-based updating strategy*. Note that, in this case, $\mathrm{Pl}_A(A) \geqslant \mathrm{Pl}(A)$ with equality holding true iff $\alpha_A = 1$ and/or $\mathrm{Pl}(A) = 1$; $\mathrm{Pl}_A(A) = 1$ iff $\mathrm{Pl}(A) = 1$.

(3) α_A violates its lower bound, i.e., $\alpha_A < \frac{1-\mathrm{Pl}(A)}{1-\mathrm{Bel}(A)}$. This yields

$$\mathrm{Bel}_A(A) > \mathrm{Pl}(A),$$
$$\mathrm{Pl}_A(A)\begin{cases} > \dfrac{[\mathrm{Pl}(A) - \mathrm{Bel}(A)] + \mathrm{Pl}(A)[1 - \mathrm{Pl}(A)]}{1 - \mathrm{Bel}(A)} & \text{for } \mathrm{Bel}(A) < 1, \\ = 1 & \text{for } \mathrm{Bel}(A) = \mathrm{Pl}(A) = 1. \end{cases} \quad (57)$$

It indicates that the BoE is willing to compromise the integrity of the originally cast evidence. A high perceived reliability associated with the incoming evidence may convince the BoE to adopt such an updating strategy.

4.3.2.1. A probabilistic interpretation. What is most interesting is that we can provide a probabilistic interpretation to the most flexible IAE-based updating strategy. To proceed, let $\mathrm{Bel}(\cdot) = \mathrm{Pl}(\cdot) \doteq P(\cdot)$. For convenience, we also assume that A and B are mutually exhaustive, i.e., $A \cup B = \Theta$. Hence

$$P(A) + P(B) - P(A \cap B) = 1. \quad (58)$$

Previously, in (48), we showed that an updating strategy comprised of a linear combination of $P(B)$ and $P(B|A)$ implies the latter to be invariant, i.e.,

$$P_A(B|A) = P(B|A) \iff \frac{P_A(A \cap B)}{P_A(A)} = \frac{P(A \cap B)}{P(A)}. \quad (59)$$

Hence let [3]

$$P_A(A) = \gamma_A P(A), \quad P_A(A \cap B) = \gamma_A P(A \cap B). \quad (60)$$

Since (58) must be true after updating as well, we have

$$\begin{aligned} P_A(B) &= 1 - P_A(A) + P_A(A \cap B) = 1 - \gamma_A[P(A) - P(A \cap B)] \\ &= 1 - \gamma_A[1 - P(B)] \qquad = (1 - \gamma_A) + \gamma_A P(B). \end{aligned} \quad (61)$$

Now, suppose m instances out of a total of M correspond to event A and the next instance corresponds to A as well. Then we may write

$$P(A) = \frac{m}{M}, \quad P_A(A) = \frac{m+1}{M+1}. \quad (62)$$

Eliminate m

$$P_A(A) = \frac{1}{M+1} + \frac{M}{M+1} P(A). \quad (63)$$

[3] This probabilistic interpretation is principally due to Professor Young [38].

Hence

$$\gamma_A = \frac{P_A(A)}{P(A)} = \frac{1 + MP(A)}{(M+1)P(A)} \quad \text{and} \quad 1 - \gamma_A = \frac{P(A) - 1}{(M+1)P(A)}. \tag{64}$$

Substitute in (61)

$$\begin{aligned} P_A(B) &= \frac{P(A) - 1}{(M+1)P(A)} + \frac{1 + MP(A)}{(M+1)P(A)} = \frac{P(A \cap B) + MP(A)P(B)}{(M+1)P(A)} \\ &= \frac{M}{M+1} P(B) + \frac{1}{M+1} P(B|A), \end{aligned} \tag{65}$$

where we have used (58). In effect, the parameters in Definition 14 are

$$\alpha_A = \frac{M}{M+1} = \frac{1 - P_A(A)}{1 - P(A)}, \quad \beta_A = \frac{1}{M+1} = \frac{P_A(A) - P(A)}{1 - P(A)}, \tag{66}$$

where (62) has been used to solve for M. In summary, the update of the probability of B conditional to the event A becomes

$$P_A(B) = \frac{1 - P_A(A)}{1 - P(A)} P(B) + \frac{P_A(A) - P(A)}{1 - P(A)} P(B|A). \tag{67}$$

Now compare with (46) and note the following correspondence:

$$P(A) \leftrightarrow \text{Bel}(A) = \text{Pl}(A), \quad P_A(A) \leftrightarrow \text{Bel}_A(A) = \text{Pl}_A(A). \tag{68}$$

This is the probabilistic interpretation of the most flexible IAE-based updating strategy we were seeking.

5. Example

To illustrate the proposed notions, consider a decision node that receives sensor data generated by magnetometers distributed throughout a battlefield. From the sensor readings it has received so far, suppose the node models its knowledge about the object located at a particular battlefield location via a BBA with the FoD $\Theta = \{metal, non\text{-}metal, empty\}$. Consider the belief, plausibility and BBA corresponding to a particular battlefield location given in Table 3.

Suppose the node then receives a new piece of evidence that the location is indeed occupied by an object (metal or non-metal). To update the BBA above, we utilize the conditioning proposition $A = \{metal, non\text{-}metal\} \Rightarrow \bar{A} = \{empty\}$. The updates corresponding to the strategy in Definition 14 appear in Table 4.

Table 3
Example 1—originally cast evidence

B	Bel(B)	Pl(B)	$m(B)$
metal	0.7	0.7	0.7
non-metal	0.1	0.2	0.1
empty	0.1	0.2	0.1
{*non-metal*, *empty*}	0.3	0.3	0.1
{*metal*, *non-metal*}	0.8	0.9	0

Table 4
Example 1—updated evidence. Note that, $\alpha_A + \beta_A = 1$

B	$\mathrm{Bel}_A(B)$	$\mathrm{Pl}_A(B)$	$m_A(B)$
metal	$0.7\alpha_A + \frac{0.7}{0.9}\beta_A$	$0.7\alpha_A + \frac{0.7}{0.8}\beta_A$	$0.7\alpha_A + \frac{0.7}{0.9}\beta_A$
non-metal	$0.1\alpha_A + \frac{0.1}{0.8}\beta_A$	$0.2\alpha_A + \frac{0.2}{0.9}\beta_A$	$0.1\alpha_A + \frac{0.1}{0.8}\beta_A$
empty	$0.1\alpha_A$	$0.2\alpha_A$	$0.1\alpha_A$
{*non-metal*, *empty*}	$0.3\alpha_A + \frac{0.1}{0.8}\beta_A$	$0.3\alpha_A + \frac{0.2}{0.9}\beta_A$	$0.1\alpha_A$
{*metal*, *non-metal*}	$0.8\alpha_A + \beta_A$	$0.9\alpha_A + \beta_A$	$\frac{0.07}{0.72}\beta_A$

Compare with the results in Sections 3.1, 3.2 and 4.1:

(1) Both {*empty*} and {*non-metal*, *empty*} properly intersect with $\bar{A}$. Hence their conditional BBAs are zero; accordingly, their updated BBAs depends only on their corresponding original BBAs.
(2) Note that {*metal*} $\in \mathscr{T}(\bar{A})$. Hence its conditional BBA does not get refined except for the re-normalization by $\mathrm{Pl}(A) = 0.9$; its updated BBA follows accordingly.
(3) $\mathscr{S}$({*non-metal*}) = $\mathscr{S}$({*metal*, *non-metal*}) = {*non-metal*, *empty*}. This is a focal element that is being annulled by conditioning. Hence neither of the propositions {*non-metal*} or {*metal*, *non-metal*} belong in $\mathscr{T}(\bar{A})$ and therefore their conditional BBAs get refined.

Next, let us consider the role $\{\alpha_A, \beta_A\}$ play in the updating mechanism. The BBAs corresponding to various strategies, together with the corresponding uncertainty intervals, are indicated in Table 5.

Note that, for the most flexible IAE-based strategy, $\alpha_A = \frac{1-\mathrm{Pl}(A)}{1-\mathrm{Bel}(A)} = \frac{1-0.9}{1-0.8} = 0.5$. In sensor information processing situations, the reliability of the incoming evidence plays a crucial role in determining when and how to update the current knowledge base. It is clear how easily $\{\alpha_A, \beta_A\}$ can accommodate this requirement

Table 5
Example 1—BBA s and uncertainty intervals corresponding to various updating strategies

Proposition B	II-based $\alpha_A = 1$	ZI-based $\alpha_A = 0$	Most flexible IAE-based $\alpha_A = 0.5$
metal	0.700 [0.700,0.700]	0.778 [0.778,0.875]	0.739 [0.739,0.787]
non-metal	0.100 [0.100,0.200]	0.125 [0.125,0.222]	0.113 [0.113,0.211]
empty	0.100 [0.100,0.200]	0.000 [0.000,0.000]	0.050 [0.050,0.100]
{*non-metal*, *empty*}	0.100 [0.300,0.300]	0.000 [0.125,0.222]	0.050 [0.213,0.261]
{*metal*, *non-metal*}	0.000 [0.800,0.900]	0.097 [1.000,1.000]	0.048 [0.900,0.950]

(1) The II-based strategy assumes the incoming evidence to be completely false and keeps its previous evidence intact.
(2) The ZI-based strategy assumes the incoming evidence to be completely reliable. The corresponding IE-based updating strategy sacrifices the integrity of the available evidence in assigning complete certainty to the proposition {*metal*, *non-metal*}, viz., incoming information that mentions the presence of an object is accepted as fact.
(3) The IAE-based strategy, on the other hand, attempts to strike a balance between these two extreme cases. It accepts the incoming evidence to the extent that the integrity of the previous evidence is not compromised. So, the belief in the proposition $\{metal, non\text{-}metal\}$ is made not to exceed its previously assigned plausibility.

6. Conclusion and future research directions

The reliability of the information being gathered and the integrity and inertia of the currently available knowledge base play crucial roles in making complex subjective decisions. This is especially true in distributed sensor networks operating in, for example, battlefield environments. In this paper, we utilize the conditional belief and plausibility notions applicable within the DS evidential reasoning framework to arrive at an evidence updating strategy to address these concerns.

First, the DS conditional notions in [27] are viewed with respect to how they impact the originally cast BBA. This viewpoint we believe is more useful since it enables one to provide a more intuitive assessment of how the conditioning proposition affects the remaining propositions. Indeed, one is now able to interpret conditioning as an annulment of the masses of all those propositions that do not *definitely* imply the conditioning proposition. Of these, only the masses of the propositions that *may* imply the conditioning proposition are redistributed to those propositions that do *definitely* imply the conditioning proposition. A characterization of these latter propositions that may 'benefit' from a proposition whose mass is being annulled is also provided.

This BBA based interpretation shows that conditioning may be viewed as a way to restrict one's viewpoint to those propositions that are in 'common' with the conditioning proposition. In other words, conditioning enhances support only for those propositions that *definitely* imply the incoming evidence while nullifying the support for all remaining propositions. It is this intuitively very appealing viewpoint that forms the basis on which the updating strategy in Section 4 has been developed. It linearly combines the available evidence with the incoming evidence conditioned to the conditioning proposition thus ensuring that masses of propositions that *may* imply the complement of the conditioning proposition are decreased while the masses of propositions that *definitely* imply the conditioning proposition are increased. We believe this to be a very sensible strategy.

In addition to this property, the proposed updating strategy also enables one to account for the reliability of incoming evidence, integrity and inertia of existing evidence and its flexibility to incoming evidence. The appeal of the proposed updating strategy lies in its ability to address these issues with ease via appropriate selection of the pair of linear combination weights which essentially 'weighs' the incoming conditional evidence against what is already available. Of particular importance is the development of a strategy to ensure that the integrity of existing evidence is not compromised. Its corresponding probabilistic interpretation we believe is quite novel and provides justification for its application.

Updating an existing knowledge base with evidence generated from different FoDs is not addressed in this work. Decision making in the presence of partial evidence generated from such non-exhaustive FoDs is a key issue encountered in several application areas. Despite its success in situations when some information essential for a probabilistic approach is unavailable and as a model for subjective human reasoning under uncertainty and representing ignorance [39], the fact that DRC in Definition 6 requires evidence to be generated from sources possessing identical FoDs has been one of its major drawbacks [40–43]. Approaches to circumvent this difficulty include the following:

- *Ignoring differences in FoDs*. In this basic approach, the evidence is assumed to discern the same frame Θ. See Fig. 4. The implication of such an assumption is that all decision processes are assumed to have access to all the information sources. In other words, it ignores the fact that each decision process may not have access to all the sources, and hence this approach cannot be considered an effective methodology. Actually, the counter-intuitive conclusions the DRC may produce under such an assumption are well documented and highlight a major drawback in the existing DS theory [44].
- *Deconditioning approach*. In this approach proposed in [25,44], the closed-world assumption made in [2] is relaxed. Consequently, the knowledge formalized in the FoD now becomes incomplete because some propositions are

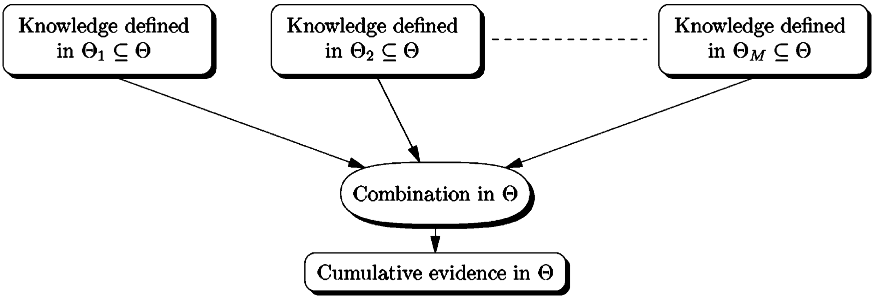

Fig. 4. Approach of ignoring differences in FoDs.

not considered. The strategy used to handle this situation assumes the existence of other sources of information that discern the missing propositions, and these sources are expected to provide the missing information. See Fig. 5. In other words, in the deconditioning approach, one supposes that the missing propositions and the propositions of the existing information sources are considered together by these other sources. Sources with adequate 'variety' in terms of propositions they consider and performance in terms of discernment of these propositions are therefore essential.

We believe that the updating strategy proposed in this paper exposes perhaps a new *conditional approach* for combining evidence when one encounters non-exhaustive FoDs. The premise of this approach is that combination of

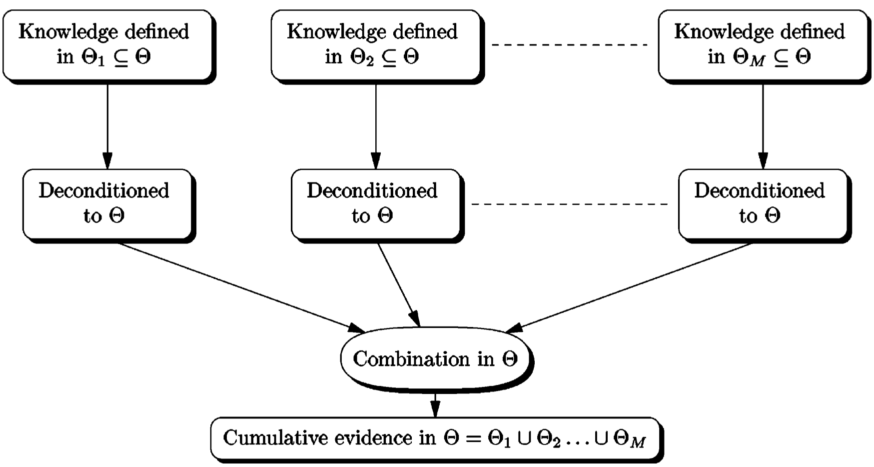

Fig. 5. Deconditioning approach.

evidence from two FoDs makes sense only when one restricts the viewpoint to those propositions belonging to their common intersection. Thus, instead of assuming 'extra' knowledge when combining non-exhaustive FoDs, one simply extracts relevant information from the smallest common sub-FoD which is then combined. This we feel provides a better representation of the uncertainty associated with the missing propositions. The envisioned approach is to first focus on those propositions belonging to the common intersection and then account for the remaining propositions. See Fig. 6.

The newly developed BBA based interpretation indicates that conditioning can be used to 'isolate' knowledge that is common to the frames; the individual evidence cast by each frame then need to be incorporated to capture the knowledge from the remaining propositions. This is essentially what a strategy composed of a linear combination of the conditional and the available evidence does—conditioning term is an indication of the propositions that are common to available evidence and incoming evidence while the other term captures the remaining propositions. Hence, we believe an appropriate generalization of the proposed updating strategy may enable both these tasks to be performed within an integrated environment.

We believe that this conditional approach, at least in certain applications, may in fact offer the better option. As an example, consider a wireless ad hoc sensor network [45] where the very limited energy reserves of nodes require each node to act as a relay of sensor information from other nodes while

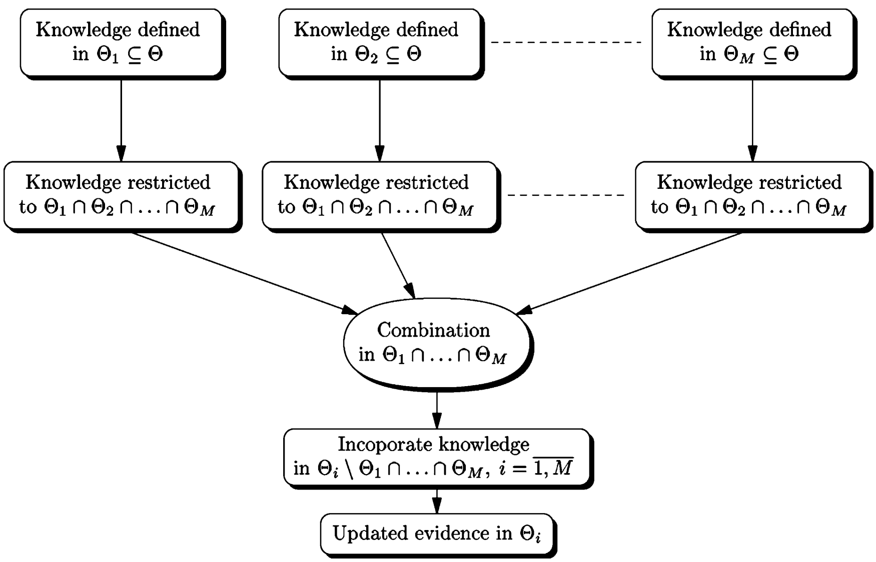

Fig. 6. Conditional approach.

generating and transmitting its own information. Can a node 'eavesdrop' into the information of other nodes it is relaying and update its own knowledge base? The conditional approach can facilitate such a refinement since it enables the node to simply concentrate on the propositions that are in common to its own frame without the need for its 'expansion'. This type of strategy would be extremely useful for, for instance, a mobile sensor which may now move and station itself at a better location to observe a particular object of interest.

Some preliminary results along the above mentioned ideas have been recently presented in [46]. Several research issues however are still under investigation. For example, an issue of critical importance that is yet to be addressed is related to the non-commutativity of the conditional [27], and hence the updating strategy. In certain applications this might in fact be desirable. For example, consider a knowledge base, such as a database of MRI images, constructed from a vast amount of evidence gathered over several years. With the arrival of a new piece of evidence, one clearly would not want to ignore the inertia of the existing database. These are issues that need careful consideration.

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